NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1425

NONLINEAR LARGE-DEFLECTION BOUNDARY-

VALUE PROBLEMS OF RECTANGULAR PLATES

By Chi-Teh Wang

Massachusetts Institute of Technology

Washington

March 1948

TECHNICAL NOTE NO. 1425

NONLINEAR LARGE-DEFLECTION BOUNDARY-

VALUE PROBLEMS OF RECTANGULAR PLATES

By Chi-Teh Wang

SUMMARY

The present report presents a theoretical analysis of an initially
flat, rectangular plate with large deflections under either normal
pressure or combined normal pressure and side thrust. As small
deflections of a flat plate are governed by a single linear equation,
large deflections introduce nonlinear terms into the conditions of
equilibrium and are governed by 2 fourth-order, second-degree, partial
differential equations. These so-called Von Kármán equations are
studied in the present report by use of the finite-difference approxi-
mations. The difference equations are solved by two methods, namely,
the method of successive approximations and the relaxation method.
Neither of these methods is new, but their application to nonlinear
problems requires new techniques.

The problem of a uniformly loaded square plate with boundary
conditions which approximate the riveted sheet-stringer panels is
solved by the method of successive approximations. The theoretical
center deflections show good agreement with the recent experimental
results obtained at the California Institute of Technology when the
deflections are of the order of the plate thickness. This agreement
perhaps suggests the range in which these Von Kármán equations are
to be applied.

Other problems of thin plates with large deflections are discussed
from the point of view of an aeronautical engineer. The boundary
conditions which approximate the various cases are formulated, and the
methods for solving these problems are outlined.

Since the method presented in the present report is general, it
may be applied to solve bending and combined bending and buckling
problems with practically any boundary conditions, and the results may
be obtained to any degree of accuracy required. Furthermore, the same
method may be applied to solve the membrane theory of the plate which
applies when the deflection is very large in comparison with the
thickness of the plate.

INTRODUCTION

The classical theory of the bending of a thin elastic plate expresses the relation between the transverse deflection of the middle surface of the plate w and the lateral loading of intensity p by the equation

$$D \nabla^4 w = p$$

where $D = \dfrac{Eh^3}{12(1 - \mu^2)}$ is the flexural rigidity of the plate. It is known that the theory is restricted in application, for on the one hand its basic assumptions can be questioned unless the plate is thin, and on the other hand it neglects an effect which must be appreciable when w has values comparable with the thickness. This is the membrane effect of curvature, whereby tension or compression in the middle surface tends to oppose or to reinforce p. The effect is negligible when w is very small, provided no stresses act initially in the plane of the middle surface; but even so, it operates when w is small because stretching the middle surface is a necessary consequence of the transverse deflection. When the deflection gets larger and larger, the membrane effect becomes more and more prominent until for very large values of w the membrane effect is predominant whereas the bending stiffness is comparatively negligible.

Small transverse displacements of a flat elastic plate are governed by a single linear equation but large displacements entail stretching of the middle surface and consequent tensions which, interacting with the curvatures, introduce nonlinear terms into the conditions of equilibrium and so make those equations no longer independent.

The large-deflection theory of flat plates is given by A. Föppl (reference 1), and the second-order terms were formulated by Theodore von Kármán in 1910 (reference 2). The amended (large-deflection) equations have been solved, however, in only a few cases (references 3 to 19) and then with considerable labor.

Essentially there are three problems concerning flat plates with large deflections. They are:

1. The bending problems, when the flat plates are subjected to lateral loading perpendicular to the plane of the plates, but no side thrust is applied in the plane of the plates

2. The buckling problems, when the plates are subjected to side thrusts in the plane of the plates but are not loaded laterally

3. The combined bending and buckling problems, when the plates are subjected to both lateral loading and side thrusts

In the case of metal airplanes, in which weight is of primary importance, the metal sheets used must be thin and the deflections of the plates are usually large in comparison with their thickness. In order to obtain the design formulas or charts for stressing such plates, the large-deflection theory must be used.

The bending problem is important in the design of seaplanes. Seaplanes are subjected to a severe impact during landing and take-off, especially on rough water. The impact must be withstood first by the bottom plating and then by a system of transverse and longitudinal members to which the bottom plating is attached, before it is transmitted to the body of the structure. The bottom should be strong enough not to washboard permanently under these impact pressures. Such washboarding is undesirable because of the increased friction between the float bottom and the water and also because of the increased aerodynamic drag in flight.

The bottom plating of seaplanes is, as a rule, subdivided into a large number of nearly rectangular areas by the transverse and longitudinal supporting ribs. Each of these areas behaves substantially like a rectangular plate under normal pressure. Bending of rectangular flat plates may therefore be used to study the washboarding of seaplane bottoms, provided the boundary conditions at the edges can be formulated just as in the seaplane.

The buckling problem is important in determining the strength of sheet-stringer panels in end compression. The use of stiffened sheet to carry compressive loads is increasingly popular in box beams for airplane wings and in other types of semi monocoque construction. Inasmuch as the sheets used as aircraft structural elements are generally quite thin, the buckling stresses of these sheet elements are necessarily low. The designer is therefore confronted with the problem of using sheet metal in the buckled or wave state and of determining the stress distribution and allowable stresses in such buckled plates.

The combined bending and buckling problem has become a problem of importance with the increasing use of wings of the stressed-skin type and the pressurized fuselage construction for high-altitude flight. During flight the wing is subjected to a pressure difference between the two sides which produces the lift. The normal pressure acts directly on the sheet covering and is then distributed to ribs and spars. At the same time the sheet panels are also subjected to a side thrust due to bending of the wing. In an airplane of pressurized fuselage construction an attempt is made to keep the pressure inside the cabin at a comfortable level for the passengers, regardless of the altitude of the airplane. Thus, there is a pressure differential across the fuselage skin with an internal pressure higher than that outside. The fuselage skin is usually subdivided into a number of rectangular curved panels by longitudinal stringers and rings. These panels are subjected to the pressure difference and side thrust resulting from bending of the fuselage. As pointed out by Niles and Newell (reference 20)

the strength of curved sheet-stringer panels can be determined approximately from the flat sheet-stringer panels. The problem is then essentially that of determining the strength of flat plates under combined lateral loading and side thrust.

Levy (reference 19) has shown that the effective width of a square plate with simply supported edges decreases with the addition of lateral pressure and that the reduction is appreciable for $\frac{pa^4}{Eh^4} > 2.25$. Therefore, a panel is unsafe if its design is based upon the side-thrust considerations only, and the study of combined loading is of great significance.

A great number of authors have studied the buckling problems, and considerable experimental work has been carried out. As a result, design formulas are available and seem to be accurate for most practical purposes. The bending problems, however, have been studied by only a few investigators, and test results (references 21 to 23) are far too scarce to justify any conclusions. The combined bending and buckling problem has been studied in only one case (reference 19), and even in this instance the results are incomplete.

Among the solutions of the large-deflection problems of rectangular plates under bending or combined bending and compression, Levy's solutions are the only ones of a theoretically exact nature. His solutions are, however, limited to a few boundary conditions and the numerical results can be obtained only after great labor.

The purpose of the present investigation is to develop a simple and yet sufficiently accurate method for the solution of the bending and the combined bending and buckling problems for engineering purposes, and this is accomplished by means of the finite-difference approximations.

Solving the partial differential equations by finite-difference equations has been accomplished previously. Solving the resulting difference equations, however, is still a problem. In the case of linear difference equations, solutions by successive approximation are always convergent and the work is only tedious. Besides, Southwell's relaxation method may be applied without too much trouble. But, in order to solve the nonlinear difference equations, the successive-approximation method cannot always be relied on because it does not always give a convergent solution. The relaxation method, since it is nothing but intelligent guessing, can be applied in only a few cases and then with great difficulties (reference 16).

A study of the finite-difference expressions of the large-deflection theory reveals that a technique can be developed by means of which the system of nonlinear difference equations can be solved with rapid convergence by successive approximation by using Crout's method of solving a system of linear simultaneous equations (reference 24). By way of illustration, a square plate under uniform normal pressure with boundary conditions approximating the riveted sheet-stringer panel is studied by this method. Nondimensional deflections and stresses are

given under various normal pressures. The results are consistent with
Levy's approximate numerical solution for ideal, simple supported plates
(reference 19) and Way's approximate solution for ideal clamped edges
(reference 15), and the center deflections check closely with the test
results by Head and Sechler (reference 23) for the ratio pa^4/Eh^4 as
large as 120. The deviation for the ratio pa^4/Eh^4 larger than 120
is probably due to the approximations employed in the derivation of
the basic differential equation.

The procedure is quite general; it may be applied to solve the
problems of rectangular plates of any length-width ratio with various
boundary conditions under either normal pressure or combined normal
pressure and side thrust.

The present investigation was originally carried out under the
direction of Professor Joseph S. Newell at the Daniel Guggenheim
Aeronautical Laboratory of the Massachusetts Institute of Technology
and was completed at Brown University, under the sponsorship and with
the financial support of the National Advisory Committee for Aeronautics,
where the author was participating in the program for Advanced Instruction
and Research in Mechanics. The author was particularly fortunate to
receive frequent advice while working on this problem from Professor
Richard von Mises of Harvard University. The author is grateful to both
Professor Newell and Professor von Mises for their many valuable
suggestions.

SYMBOLS

a, b	length and width of plate, respectively
h	thickness of plate
x, y, z	coordinates of a point in plate
u, v	horizontal displacements of points in middle surface in x- and y-directions, respectively (nondimensiona. forms are ua/h^2, va/h^2, respectively)
w	deflection of middle surface from its initial plane (nondimensional form is w/h)
p	normal load on plate per unit area (nondimensional form is pa^4/Eh^4)
E, μ	Young's modulus and Poisson's ratio, respectively
D	flexural rigidity of plate $\left(\dfrac{Eh^3}{12(1 - \mu^2)} \right)$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}$$

σ_x', σ_y', τ_{xy}' membrane stresses in middle surface (nondimensional forms are $\sigma_x'a^2/Eh^2$, $\sigma_y'a^2/Eh^2$, and $\tau_{xy}'a^2/Eh^2$, respectively)

σ_x'', σ_y'', τ_{xy}'' extreme-fiber bending and shearing stresses (nondimensional forms are $\sigma_x''a^2/Eh^2$, $\sigma_y''a^2/Eh^2$, and $\tau_{xy}''a^2/Eh^2$, respectively)

ϵ_x', ϵ_y', γ_{xy}' membrane strains in middle surface (nondimensional forms are $\epsilon_x'a^2/h^2$, $\epsilon_y'a^2/h^2$, and $\gamma_{xy}'a^2/h^2$, respectively)

ϵ_x'', ϵ_y'', γ_{xy}'' extreme-fiber bending and shearing strains (nondimensional forms are $\epsilon_x''a^2/h^2$, $\epsilon_y''a^2/h^2$, and $\gamma_{xy}''a^2/h^2$, respectively)

F stress function (nondimensional form is F/Eh^2)

Δ, Δ^2,..., Δ^n first-, second-, ..., to nth-order differences, respectively

Δ_x, Δ_y first-order differences in x- and y-directions, respectively

FUNDAMENTAL DIFFERENTIAL EQUATIONS

The thickness of the plate is assumed small compared with its other dimensions. The middle plane of the plate is taken to coincide with the xy-plane of the coordinate system and to be a plane of elastic symmetry. After bending, the points of the middle plane are displaced and lie on some surface which is called the middle surface of the plate. The displacement of a point of the middle plane in the direction of the z-axis w is called the deflection of the given point of the plate.

Consider the case in which the deflections are large in comparison with the thickness of the plate but, at the same time, are small enough to justify the following assumptions:

1. Lines normal to the middle surface before deformation remain normal to the middle surface after deformation.

2. The normal stress σ_z perpendicular to the faces of the plate is negligible in comparison with the other normal stresses.

In order to investigate the state of strain in a bent plate, it is supposed that the middle surface is actually deformed and that the deflections are no longer small in comparison with the thickness of the plate but are still small as compared with the other dimensions.

Under these assumptions, the following fundamental partial differential equations governing the deformation of thin plates can be derived from the compatibility and equilibrium conditions:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} + \frac{h}{D}\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)$$

where $D = \dfrac{Eh^3}{12(1 - \mu^2)}$, the median-fiber stresses are

$$\sigma_x{}' = \frac{\partial^2 F}{\partial y^2}$$

$$\sigma_y{}' = \frac{\partial^2 F}{\partial x^2}$$

$$\tau_{xy}{}' = -\frac{\partial^2 F}{\partial x \partial y}$$

and the median-fiber strains are

$$\epsilon_x{}' = \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right)$$

$$\epsilon_y{}' = \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2}\right)$$

$$\gamma_{xy}' = -\frac{2(1 + \mu)}{E} \frac{\partial^2 F}{\partial x \, \partial y}$$

The extreme-fiber bending and shearing stresses are

$$\sigma_x'' = -\frac{Eh}{2(1 - \mu^2)} \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma_y'' = -\frac{Eh}{2(1 - \mu^2)} \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right)$$

$$\tau_{xy}'' = -\frac{Eh}{2(1 + \mu)} \frac{\partial^2 w}{\partial x \, \partial y}$$

These expressions can be made nondimensional by writing

$$F' = \frac{F}{h^2 E} \qquad\qquad x' = \frac{x}{a}$$

$$w' = \frac{w}{h} \qquad\qquad y' = \frac{y}{a}$$

$$p' = \frac{pa^4}{Eh^4} \qquad\qquad \epsilon' = \epsilon \left(\frac{a}{h} \right)^2$$

$$\sigma' = \frac{\sigma}{E} \left(\frac{a}{h} \right)^2$$

where a is the smaller side of the rectangular plate.

The differential equations then become

$$\frac{\partial^4 F'}{\partial x'^4} + 2\frac{\partial^4 F'}{\partial x'^2 \, \partial y'^2} + \frac{\partial^4 F'}{\partial y'^4} = \left(\frac{\partial^2 w'}{\partial x' \, \partial y'} \right)^2 - \frac{\partial^2 w'}{\partial x'^2} \frac{\partial^2 w'}{\partial y'^2}$$

$$\frac{\partial^4 w'}{\partial x'^4} + 2\frac{\partial^4 w'}{\partial x'^2 \, \partial y'^2} + \frac{\partial^4 w'}{\partial y'^4} = 12(1 - \mu^2)p' + 12(1 - \mu^2)\left(\frac{\partial^2 F'}{\partial y'^2} \frac{\partial^2 w'}{\partial x'^2} \right.$$

$$\left. + \frac{\partial^2 F'}{\partial x'^2} \frac{\partial^2 w'}{\partial y'^2} - 2\frac{\partial^2 F'}{\partial x' \, \partial y'} \frac{\partial^2 w'}{\partial x' \, \partial y'} \right)$$

If $\mu^2 = 0.1$, which value is characteristic of aluminum alloys, and the primes are dropped, the partial differential equations in nondimensional form are

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = \left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{1}$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 10.8p + 10.8\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right.$$

$$\left. + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \, \partial y}\frac{\partial^2 w}{\partial x \, \partial y}\right) \tag{2}$$

The nondimensional median-fiber stresses are

$$\left.\begin{array}{l} \sigma_x{}' = \dfrac{\partial^2 F}{\partial y^2} \\[2mm] \sigma_y{}' = \dfrac{\partial^2 F}{\partial x^2} \\[2mm] \tau_{xy}{}' = -\dfrac{\partial^2 F}{\partial x \, \partial y} \end{array}\right\} \tag{3}$$

and the nondimensional median-fiber strains are

$$\left.\begin{array}{l} \epsilon_x{}' = \dfrac{\partial^2 F}{\partial y^2} - \mu\dfrac{\partial^2 F}{\partial x^2} \\[2mm] \epsilon_y{}' = \dfrac{\partial^2 F}{\partial x^2} - \mu\dfrac{\partial^2 F}{\partial y^2} \\[2mm] \gamma_{xy}{}' = -2(1 + \mu)\dfrac{\partial^2 F}{\partial x \, \partial y} \end{array}\right\} \tag{4}$$

The nondimensional extreme-fiber bending and shearing stresses are

$$\sigma_x" = -\frac{1}{2(1 - \mu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y" = -\frac{1}{2(1 - \mu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} \quad (5)$$

$$\tau_{xy}" = -\frac{1}{2(1 + \mu)}\frac{\partial^2 w}{\partial x\, \partial y}$$

FORMULATION OF BOUNDARY CONDITIONS

The governing differential equations are 2 fourth-order simultaneous partial differential equations in two variables. In order to obtain a unique solution in the case of rectangular plates, there must be four given boundary conditions at each edge.

Before proceeding to the actual case, two theoretical boundary conditions may be mentioned:

1. Simply supported plates, that is, plates having edges that can rotate freely about the supports and can move freely along the supports

2. Clamped or built-in plates, that is, plates having edges that are clamped rigidly against rotation about the supports and at the same time are prevented from having any displacements along the supports

Actually, it is to be expected that neither of these conditions will be fulfilled exactly in a structure.

The bending problem will be considered next, in which the bottom plating of a seaplane is to be studied. The behavior of the sheet approximates that of an infinite sheet supported on a homogeneous elastic network with rectangular fields of the same rigidity as the supporting framework of the seaplane.

Because of the symmetry of the rectangular fields, the displacement in the plane of the sheet and the slope of the sheet relative to the plane of the network must be zero wherever the sheet passes over the center line of each supporting beam. Each rectangular field will therefore behave as a rectangular plate clamped along its four edges on supports that are rigid enough in the plane of the sheet to prevent their displacement in that plane. At the same time these supports must have a rigidity normal to the plane of the sheet equal to that of the actual supports in the flying-boat bottom.

The rigidity of the supports will lie somewhere between the unattainable extremes of zero rigidity and infinite rigidity. The extreme of infinite rigidity normal to the plane of the sheet is one that may be approximated in actual designs. It can be shown that the stress distribution in such a fixed-edge plate will, in most cases, be less favorable than the stress distribution in the elastic-edge plate. The strength of plates obtained from the theory will therefore be on the safe side if applied in flying-boat design. Reference might be made in this connection to a paper by Mesnager (reference 25), in which it is shown that a rectangular plate with elastic edges of certain flexibility will be less highly stressed than a clamped-edge plate. This difference in stress may also be clearly seen by comparing the extreme-fiber-stress calculations by Levy (reference 19) and Way (reference 15) for simply supported plates and clamped plates.

The impact pressure on a flying-boat bottom in actual cases, however, is not even approximately uniform over a portion of the sheet covering several rectangular fields. Usually one rectangular panel of the bottom plating would resist a higher impact pressure than the surrounding panels, and the sheet is supported on beams of torsional stiffness insufficient to develop large moments along the edges. The high bending stresses at the edges characteristic of rigidly clamped plates would then be absent. In order to approximate this condition, the plate may be assumed to be simply supported so that it is free to rotate about the supports. At the same time the riveted joints prevent it from moving in the plane of the plate along and perpendicular to the supports. According to the same considerations as in the case of rigidly clamped edges, the result would be on the safe side. This case has never before been discussed and the study of such a problem seems to be of importance.

For the combined bending and buckling problems the same considerations will hold. It is evident, however, that as soon as the side thrust is applied, there are displacements perpendicular to the supported edges in the plane of the plate. Gall (reference 26) has found that a stiffener attached to a flat sheet carrying a compressive load contributed approximately the same elastic support to the sheet as was required to give a simply supported edge (see also reference 20, p. 327). In combined bending and compression problems, therefore, it seems also important to study the ideal simply supported plates. The analytical expressions for these boundary conditions are formulated in the following discussion.

Simply Supported Edge

If the edge $y = 0$ of the plate is simply supported, the deflection w along this edge must be zero. At the same time this edge can rotate freely with respect to the x-axis; that is, there is no bending moment M_y along this edge. In this case, the analytical formulation of the physical boundary conditions is

$$(w)_{y=0} = 0$$

$$\left. \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \right|_{y=0} = 0 \quad \Bigg\} \tag{6}$$

Similarly, if the edge $x = 0$ of the plate is simply supported, the boundary conditions are

$$(w)_{x=0} = 0$$

$$\left. \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \right|_{x=0} = 0$$

Since $w = 0$ along $y = 0$, $\partial w/\partial x$ and $\partial^2 w/\partial x^2$ must be zero also. The boundary conditions can therefore be written as

$$(w)_{y=0} = 0$$

$$\left. \left(\frac{\partial^2 w}{\partial y^2} \right) \right|_{y=0} = 0 \quad \Bigg\} \tag{7}$$

Similarly, on the edge $x = 0$,

$$(w)_{x=0} = 0$$

$$\left. \left(\frac{\partial^2 w}{\partial x^2} \right) \right|_{x=0} = 0$$

If the plate has ideal simply supported edges, it must be free to move along the supported edges in the plane of the plate; that is, the shearing stress along the edges in the plane of the plate is zero. Analytically,

$$\left(\tau_{xy}' \right)_{y=0} = 0$$

$$\left(\tau_{xy}' \right)_{x=0} = 0$$

or

$$\left(\frac{\partial^2 F}{\partial x\ \partial y}\right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial x\ \partial y}\right)_{x=0} = 0$$

One more boundary condition is required to solve the plate problems uniquely, and this may be obtained by specifying either the normal stresses or the displacements along the edges.

For a plate having zero-edge compression, the normal stresses along the edges are zero. That is,

$$\left(\sigma_x{}'\right)_{x=0} = 0$$

$$\left(\sigma_y{}'\right)_{y=0} = 0$$

or

$$\left.\begin{array}{c}\left(\dfrac{\partial^2 F}{\partial y^2}\right)_{x=0} = 0 \\[20pt] \left(\dfrac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0\end{array}\right\} \tag{8}$$

The strain in the median plane is

$$\epsilon_x{}' = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_y{}' = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

Therefore

$$\frac{\partial u}{\partial x} = \epsilon_x' - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\frac{\partial v}{\partial y} = \epsilon_y' - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

and the displacement of the edges in the x-direction is

$$u = \int_{y=\text{Constant}}\left[\epsilon_x' - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dx$$

while the displacement of the edges in the y-direction is

$$v = \int_{x=\text{Constant}}\left[\epsilon_y' - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]dy$$

The addition of side thrust may be expressed in terms of the change in displacement of the edges.

If ϵ_x' and ϵ_y' are expressed in terms of the stress function F,

$$\left.\begin{array}{l} u = \int_{y=\text{Constant}}\left[\dfrac{\partial^2 F}{\partial y^2} - \mu\dfrac{\partial^2 F}{\partial x^2} - \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\right]dx \\[4em] v = \int_{x=\text{Constant}}\left[\dfrac{\partial^2 F}{\partial x^2} - \mu\dfrac{\partial^2 F}{\partial y^2} - \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2\right]dy \end{array}\right\} \qquad (9)$$

Clamped or Built-In Edge

If an edge of a plate is clamped, the deflection along this edge is zero, and the plane tangent to the deflected middle surface along this edge coincides with the initial position of the middle plane of the plate.

If the x-axis coincides with the clamped edge, the boundary
conditions are

$$\left.\begin{array}{r} (w)_{y=0} = 0 \\[2em] \left(\dfrac{\partial w}{\partial y}\right)_{y=0} = 0 \end{array}\right\} \tag{10}$$

If the y-axis coincides with the clamped edge, the boundary
conditions are

$$(w)_{x=0} = 0$$

$$\left(\dfrac{\partial w}{\partial x}\right)_{x=0} = 0$$

If the edge is clamped rigidly against any displacement along its
support, the strain in the median fibers must be zero along that edge.
The boundary conditions are

$$\left(\epsilon_y{}'\right)_{x=0} = 0$$

$$\left(\epsilon_x{}'\right)_{y=0} = 0$$

or

$$\left.\begin{array}{r} \left(\dfrac{\partial^2 F}{\partial x^2} - \mu\dfrac{\partial^2 F}{\partial y^2}\right)_{x=0} = 0 \\[2em] \left(\dfrac{\partial^2 F}{\partial y^2} - \mu\dfrac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0 \end{array}\right\} \tag{11}$$

The one additional condition required is again furnished by
specifying the displacements along the edges as in equation (9).

Riveted Panel with Normal Pressure Greater than That of

Surrounding Panels

The boundary conditions which would approximate this situation are, if $y = 0$ is one of the edges,

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0 \tag{12}$$

$$\int_{x=\text{Constant}} \left[\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2} - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] dy = 0$$

The first two expressions are those of simply supported edges, the third one gives the condition of zero strain along the supports, and the last one specifies that the displacement along the edge is zero.

REVIEW OF PREVIOUS WORK

The large-deflection theory of flat plates is given by A. Föppl (reference 1), and the difficulty of solving the nonlinear equations has been noted by Theodore von Kármán (reference 2). The earliest attempt to deal with these differential equations was, perhaps, made by H. Hencky (references 3 and 4), who devised an approximate method of solution for circular and square plates when the deflection is very large, the bending stiffness being then negligible. Following the same procedure, Kaiser (reference 5) solved the case of a simply supported plate with zero edge compression under lateral loading. His theoretical result checked closely with his experimental data.

In the case of circular plates with large deflections, because of the radial symmetry, the two fundamental partial differential equations

which contain the linear biharmonic differential operator and quadratic
terms in the second derivatives can be reduced to a pair of ordinary
nonlinear differential equations, each of the second order. For both
the bending and the buckling problems, exact solutions are available
(references 8 to 12). The bending problem has been solved approximately
by Nadai (reference 6) and Timoshenko (reference 7) and exactly by
Way (reference 8) when the plate is under lateral pressure and edge
moment. Way gave a power-series solution for a rather large range of
applied load. The buckling problem has been solved by Federhofer
(reference 9) and Friedrichs and Stoker (references 10 to 12). Federhofer
gave the solution for both simply supported and clamped edges which
yields accurate results up to values of N of about 1.25, where N is
the ratio of the pressure applied at the edge to the lowest critical
or Euler's pressure at which the buckling just begins. Friedrichs
and Stoker gave a complete solution for the simply supported circular
plate for N up to infinity. To cover this range, they employ three
methods. Each of the three methods is suitable for a particular range
of values of N: namely, the perturbation method for low N, the
power-series method for intermediate N, and the asymptotic solution
for N approaching infinity. There is no solution, however, for
the case of circular plates under combined lateral pressure and edge
thrust.

The exact solution for a thin, infinitely long, rectangular
strip with clamped or simply supported edges was obtained by Boobnoff
and Timoshenko (references 13 and 27), and the other cases were discussed
by Prescott (reference 14), Way (reference 15), Green and Southwell
(reference 16), Levy (references 17 and 19), and Levy and Greenman
(reference 18).

Prescott gives an approximate solution for the simply supported
plate with no edge displacement; however, Prescott's approximation
is rather rough. Way presented a better approximate solution for the
clamped plates by using the Ritz energy method. Kaiser (reference 5)
transformed the differential equations into finite-difference equations
and solved them by the trial-and-error method. Green and Southwell
extended the finite-difference study into finer divisions and solved
the difference equations by means of the relaxation method.

Levy (reference 19) gives a general solution for simply supported
plates, and numerical solutions are given for square and rectangular
plates with a width-span ratio of 3 to 1 under some combined lateral
and side loading conditions. Levy and Greenman (references 17 and 18)
extended this solution for simply supported edges to clamped edges.
Their conditions are, however, limited to the case in which the edge
supports are assumed to clamp the plate rigidly against rotations and
displacements normal to the edge but to allow displacements parallel
to the edge. They presented a numerical solution for square and
rectangular plates with a width-span ratio of 3 to 1 under lateral
pressure.

In summary the problem of rectangular plates with large deflections has been solved by three methods: namely, the energy method, the finite-difference-equations method, and the Fourier series method. These methods are briefly outlined in the following paragraphs.

Energy Method

The method of attack used by Way (reference 15) is the Ritz energy method. Expressions are assumed for the three displacements in the form of algebraic polynomials satisfying the boundary conditions; then, by means of minimizing the energy with respect to the coefficients, a system of simultaneous equations is obtained, the solution of which gives these coefficients.

The energy expression for plates with large deflection is

$$V = \iint \left\{ \frac{(\nabla^2 w)^2}{2} - qw + 6\left[u_x{}^2 + u_x w_x{}^2 + v_y{}^2 + v_y w_y{}^2 \right.\right.$$

$$+ \frac{1}{4}(w_x{}^2 + w_y{}^2) + 2\mu \left(u_x v_y + v_y \frac{w_x{}^2}{2} + u_x \frac{w_y{}^2}{2} \right)$$

$$\left.\left. + \frac{1 - \mu}{2} (u_y{}^2 + 2u_y v_x + v_x{}^2 + 2u_y w_x w_y + 2v_x w_x w_y) \right] \right\} dx\, dy \quad (13)$$

where u and v are the nondimensional horizontal displacements and w is the nondimensional vertical displacement, $q = \dfrac{pa^4}{16Dh}$, and the subscripts indicate partial differentiation. In order that u, v, and w satisfy the boundary conditions for clamped edges, Way assumes (fig. 1):

$$\left. \begin{array}{l} u = (1 - x^2)(\beta^2 - y^2)x(b_{00} + b_{02}y^2 + b_{20}x^2 + b_{22}x^2y^2) \\[2ex] v = (1 - x^2)(\beta^2 - y^2)y(c_{00} + c_{02}y^2 + c_{20}x^2 + c_{22}x^2y^2) \\[2ex] w = (1 - x)^2(\beta^2 - y^2)^2(a_{00} + a_{02}y^2 + a_{20}x^2) \end{array} \right\} \quad (14)$$

where $\beta = \dfrac{b}{a}$; u, v, w are positive in the positive directions of x, y, z, respectively; and a_{ij}, b_{ij}, c_{ij} are numerical constants to be determined later. For convenience, i is taken to be the same as the power of x, and j that of y.

When V is minimized with respect to the coefficients a_{ij}, b_{ij}, and c_{ij}, 11 simultaneous equations corresponding to the 11 constants, are obtained as follows:

$$\frac{\partial V}{\partial a_{00}} = 0; \quad \frac{\partial V}{\partial a_{02}} = 0; \quad \frac{\partial V}{\partial a_{20}} = 0 \tag{15}$$

$$\frac{\partial V}{\partial b_{00}} = 0; \quad \frac{\partial V}{\partial b_{02}} = 0; \quad \frac{\partial V}{\partial b_{20}} = 0; \quad \frac{\partial V}{\partial b_{22}} = 0 \tag{16}$$

$$\frac{\partial V}{\partial c_{00}} = 0; \quad \frac{\partial V}{\partial c_{02}} = 0; \quad \frac{\partial V}{\partial c_{20}} = 0; \quad \frac{\partial V}{\partial c_{22}} = 0 \tag{17}$$

These equations are not linear in the constants. The first three equations (equation (15)) will contain terms of the third degree in the a's. Equations (16) and (17) are linear in the b's and c's and quadratic in the a's. Way solved equations (16) and (17) for b's and c's, respectively, in terms of a's and then substituted these expressions in equation (15). There then are left three equations of third degree involving the a's alone. These were solved by Way by successive approximations.

Way gives the numerical solutions for cases for which $\beta = 1$, 1.5, and 2, for $\mu = 0.3$ up to $q = 210$. Since he assumed the displacements to be polynomials in x and y of finite number of terms, his solutions are essentially approximate. By comparing with Boobnoff's exact solution for the infinite plate, Way estimated that the error of his solution for $\beta = 2$ is about 10 percent on the conservative side.

Finite-Difference Methods of Solution

Kaiser writes the nondimensional Von Kármán equations as follows:

$$\nabla^2 S = \left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} = -K$$

$$\nabla^2 F = S$$

$$2\frac{\partial^2 F}{\partial x \, \partial y} \frac{\partial^2 w}{\partial x \, \partial y} - \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} = p_G$$

$$\frac{1}{12(1 - \mu^2)}\nabla^2 M = p - p_G$$

$$\nabla^2 w = M$$

(18)

and then transforms these five equations into finite-difference equations. His procedure is to assume w's at all the points and then to solve for S's, F's, M's, and w's. If the calculated w's do not check with the assumed ones, he assumes a new set of w's and repeats the process. The work which this involves is very tedious. In fact, as will be pointed out later, when the usual method of successive approximations is used, the process is actually divergent. Kaiser solved the simply supported square plate with zero edge compression under a uniform lateral pressure of $\frac{pa^4}{Eh^4} = 118.72$. His numerical solution checked with his experimental results with good accuracy.

Southwell and Green solved four examples of the problem by means of a technique based on the relaxation method. The fundamental requirements for use of the relaxation technique are a simple finite-difference pattern of the variables and a simple expression of the boundary conditions. In using this, Southwell and Green expressed the differential equations in terms of the displacements u, v, and w, which then gave simple boundary conditions. Instead of using exact relaxation patterns, they worked with the patterns which are given by the linear terms of the differential equations and made corrections from time to time, the nonlinear terms being combined with the "residue."

$$\nabla^4 w = \frac{p}{D} + \frac{3}{h^2} \left\{ \frac{\partial^2 w}{\partial x^2} \left[\frac{\partial u}{\partial x} + \mu \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial w}{\partial y}\right)^2 \right] + \frac{\partial^2 w}{\partial y^2}\left[\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x} \right. \right.$$

$$\left. \left. + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial w}{\partial x}\right)^2 \right] + (1 - \mu)\ \frac{\partial^2 w}{\partial x\, \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right) \right\}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1 - \mu}{1 + \mu}\nabla^2 u + \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1 - \mu}{1 + \mu}\frac{\partial w}{\partial x}\nabla^2 w = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1 - \mu}{1 + \mu}\nabla^2 v + \frac{1}{2}\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1 - \mu}{1 + \mu}\frac{\partial w}{\partial y}\nabla^2 w = 0$$

$$\left. \right\} (19)$$

It is readily seen that, in order to obtain a simple expression for the boundary conditions, not only is the number of the partial differential equations increased from two to three, but also the form of the terms involved becomes more complicated and the number of terms is increased. This technique proves very laborious in practice.

Equation (19), expressing conditions of equilibrium, could have been derived by minimizing the total potential energy V, which is given by the expression

$$\frac{L^2}{h^2}\frac{V}{D} = I_1 + I_2 + I_3 \qquad (20)$$

where

$$I_1 = \frac{1}{2}\iint \left(\nabla^2 w\right)^2 dx\ dy$$

$$I_2 = \frac{3}{2}\iint \left(e_{xx}^2 + e_{yy}^2 + 2\mu e_{xx}e_{yy} + \frac{1 - \mu}{2}\ e_{xy}^2 \right) dx\ dy$$

and

$$I_3 = -\alpha \iint w\ dx\ dy$$

where α is the lateral loading.

The relaxation technique consists first in assuming a set of answers and then changing them according to the relaxation pattern and boundary conditions. To obtain a more rapid convergence, Southwell and Green multiplied the given values of w by k and substituted them into the energy expression to obtain

$$\frac{L^2}{h^2} \frac{V}{D} = k^2 I_1 + k^4 I_2 + \alpha k I_3 \qquad (21)$$

which was then minimized with respect to k; that is, by setting $\frac{\partial V}{\partial k} = 0$ to give

$$2k I_1 + 4k^3 I_2 - \alpha I_3 = 0 \qquad (22)$$

From the third-order equation (equation (22)), k can be obtained and a set of values for w which are closer to the true values can be derived from values of k.

Fourier Series Methods of Solution

Levy and Greenman obtained general solutions of the rectangular plates (fig. 2) under combined bending and side thrust with large deflections by means of Fourier series. Their approach to these problems is given in the following discussion.

Simply supported rectangular plates.- In order to satisfy the boundary conditions, w is assumed to be given by the Fourier series

$$w = \sum_{m=1,2,3}^{\infty} \sum_{n=1,2,3}^{\infty} W_{m,n} \sin m \frac{\pi x}{a} \sin n \frac{\pi y}{b} \qquad (23)$$

The normal pressure may be expressed as a Fourier series

$$p_z = \sum_{r=1,2,3}^{\infty} \sum_{s=1,2,3}^{\infty} p_{r,s} \sin r \frac{\pi x}{a} \sin s \frac{\pi y}{b} \qquad (24)$$

For the compatibility equation to be satisfied, F must be given by

$$F = -\frac{\bar{p}_x y^2}{2} - \frac{\bar{p}_y x^2}{2} + \sum_{p=0,1,2}^{\infty} \sum_{q=0,1,2}^{\infty} b_{p,q} \cos p \frac{\pi x}{a} \cos q \frac{\pi y}{b} \qquad (25)$$

where $\bar{p}_x$ and $\bar{p}_y$ are constants equal to the average membrane pressure in the x- and y-directions, respectively, and where

$$b_{p,q} = \frac{E}{4\left(p^2 \frac{b}{a} + q^2 \frac{a}{b}\right)^2}(B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9) \qquad (26)$$

and

$$B_1 = \sum_{k=1}^{p-1} \sum_{t=1}^{q-1} \left[kt(p-k)(q-t) - k^2(q-t)^2 \right] w_{k,t} w_{p-k,q-t}$$

if $q \neq 0$ and $p \neq 0$.

$B_1 = 0$ if $q = 0$ or $p = 0$.

$$B_2 = \sum_{k=1}^{\infty} \sum_{t=1}^{q-1} \left[kt(k+p)(q-t) + k^2(q-t)^2 \right] w_{k,t} w_{k+p,q-t}$$

if $q \neq 0$.

$B_2 = 0$ if $q = 0$.

$$B_3 = \sum_{k=1}^{\infty} \sum_{t=1}^{q-1} \left[(k+p)kt(q-t) + (k+p)^2(q-t)^2 \right] w_{k+p,t} w_{k,q-t}$$

if $q \neq 0$ and $p \neq 0$.

$B_3 = 0$ if $q = 0$ or $p = 0$.

$$B_4 = \sum_{k=1}^{p-1} \sum_{t=1}^{\infty} \left[kt(p-k)(t+q) + k^2(t+q)^2 \right] w_{k,t} w_{p-k,t+q}$$

if $p \neq 0$.

$B_4 = 0$ if $p = 0$.

$$B_5 = \sum_{k=1}^{p-1} \sum_{t=1}^{\infty} \left[kt(t+q)(p-k) + k^2 t^2 \right] w_{k,t+q} w_{p-k,t}$$

if $p \neq 0$ and $q \neq 0$.

$B_5 = 0$ if $p = 0$ or $q = 0$.

$$B_6 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[kt(k+p)(t+q) - k^2(t+q)^2 \right] w_{k,t} w_{k+p,t+q}$$

if $q \neq 0$.

$B_6 = 0$ if $q = 0$.

$$B_7 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[kt(t+q)(k+p) - k^2 t^2 \right] w_{k,t+q} w_{k+p,t}$$

if $q \neq 0$ and $p \neq 0$.

$B_7 = 0$ if $p = 0$ or $q = 0$.

$$B_8 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[kt(k+p)(t+q) - (k+p)^2(t+q)^2 \right] w_{k+p,t} w_{k,t+q}$$

$$\text{if} \quad q \neq 0 \quad \text{and} \quad p \neq 0.$$

$$B_8 = 0 \quad \text{if} \quad p = 0 \quad \text{or} \quad q = 0.$$

$$B_9 = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} \left[(k + p)(t + q)kt - (k + p)^2 t^2 \right] w_{k+p,\,t+q} w_{k,\,t}$$

$$\text{if} \quad p \neq 0.$$

$$B_9 = 0 \quad \text{if} \quad p = 0.$$

The equilibrium equation is satisfied if

$$p_{r,s} = Dw_{r,s}\left(r^2\frac{\pi^2}{a^2} + s^2\frac{\pi^2}{b^2}\right)^2 - \bar{p}_x h w_{r,s} r^2 \frac{\pi^2}{a^2} - \bar{p}_y h w_{r,s} s^2 \frac{\pi^2}{b^2}$$

$$+ \frac{h\pi^4}{4a^2 b^2}\left\{ -\sum_{k=1}^{r}\sum_{t=1}^{s}(s-t)k - (r-k)t^2 b_{r-k,s-t}w_{k,t}\right.$$

$$-\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\left[t(k+r) - k(t+s)\right]^2 b_{k,t}w_{k+r,t+s}$$

$$+\sum_{k=0}^{\infty}\sum_{t=1}^{\infty}\left[(k+r)(t+s) - kt\right]^2 b_{k,t+s}w_{k+r,t}$$

$$+\sum_{k=1}^{\infty}\sum_{t=0}^{\infty}\left[tk - (k+r)(t+s)\right]^2 b_{k+r,t}w_{k,t+s}$$

$$-\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\left[(t+s)k - (k+r)t\right]^2 b_{k+r,t+s}w_{k,t}$$

$$-\sum_{k=1}^{r}\sum_{t=0}^{\infty}\left[tk + (r-k)(t+s)\right]^2 b_{r-k,t}w_{k,t+s}$$

$$+\sum_{k=1}^{r}\sum_{t=1}^{\infty}\left[(t+s)k + (r-k)t\right]^2 b_{r-k,t+s}w_{k,t}$$

$$-\sum_{k=0}^{\infty}\sum_{t=1}^{s}\left[(s-t)(k+r) + tk\right]^2 b_{k,s-t}w_{k+r,t}$$

$$\left.+\sum_{k=1}^{\infty}\sum_{t=1}^{s}\left[(s-t)k + t(k+r)\right]^2 b_{k+r,s+t}w_{k,t}\right\} \qquad (27)$$

When the lateral pressure is given, $p_{r,s}$ can be determined. Equation (27) represents a doubly infinite family of equations. In each of the equations of the family, the coefficients $b_{p,q}$ may be replaced by their values as given by equation (26). The resulting equations will involve the known normal pressure coefficients $p_{r,s}$, the cubes of the deflection coefficients $w_{m,n}$, and the average membrane pressures in the x- and the y-directions, $\bar{p}_x$ and $\bar{p}_y$, respectively. Values of $\bar{p}_x$ and $\bar{p}_y$ can be determined from the conditions that the plates are either subjected to known edge compressions or known edge displacements. The number of these equations is equal to the number of unknown deflection coefficients $w_{m,n}$.

The procedure now is, with the known values of $p_{r,s}$, to assume $w_{1,1}$ and to solve the other coefficients by successive approximation. However, the work involved is tremendous, and it is very easy to make mistakes. As illustrated by Levy in a relatively simple case of a square plate, if six deflection coefficients are used, then each equation contains 60 third-order terms. And for each given applied normal pressure these six 60-term, third-order equations must be solved by successive approximations.

Clamped rectangular plates.- Levy and Greenman solved the case of the clamped rectangular plate by assuming that the edges are clamped rigidly against rotations and displacements normal to the edges but are permitted to move freely parallel to the edges.

The required edge moments m_x and m_y are replaced by an auxiliary pressure distribution $p_a(x,y)$ near the edges of the plate. The auxiliary pressure can be expressed as a Fourier series as follows:

$$p_a(x,y) = \sum_{r=1,3,5}^{\infty} \frac{4\pi m_y}{a^2} r \sin \frac{r\pi x}{a} + \sum_{s=1,3,5}^{\infty} \frac{4\pi m_x}{b^2} s \sin \frac{s\pi y}{b} \qquad (28)$$

By writing m_x and m_y as Fourier series, where k_s and k_r are coefficients to be determined,

$$\left.\begin{aligned} m_x &= \frac{4a^2}{\pi^3} p \sum_{r=1,3,5}^{\infty} k_r \sin \frac{r\pi x}{a} \\[2em] m_y &= \frac{4b^2}{\pi^3} p \sum_{s=1,3,5}^{\infty} k_s \sin \frac{s\pi y}{b} \end{aligned}\right\} \qquad (29)$$

Inserting equation (29) into equation (28) gives

$$p_a(x,y) = \left(\frac{4}{\pi}\right)^2 p \sum_{r=1,3,5}^{\infty} \sum_{s=1,3,5}^{\infty} (rk_s + sk_r) \sin\frac{r\pi x}{a} \sin\frac{s\pi y}{b} \qquad (30)$$

On combining the auxiliary pressure $p_a(x,y)$ with the normal pressure p_z, equation (24), the following equation is obtained:

$$p_c(x,y) = \sum_{r=1,2,3}^{\infty} \sum_{s=1,2,3}^{\infty} p_{r,s} \sin r\frac{\pi x}{a} \sin s\frac{\pi y}{b} \qquad (31)$$

where

$$p_{r,s} = \left(\frac{4}{\pi}\right)^2 (rpk_s + spk_r) + p_{r,s}' \qquad (32)$$

Since the edge moments m_x and m_y have been replaced by the auxiliary pressure distribution $p_a(x,y)$, the general solution for the simply supported rectangular plate (equations (23) to (27)) can be applied to clamped plates, and the remaining problem is to determine the values of k_s and k_r. These values are obtained by use of the boundary condition that the slope at the edges of the plate is zero. Equating to zero the normal slopes along the edges gives

$$\left.\left(\frac{\partial w}{\partial x}\right)\right|_{x=0,x=a} = 0 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{m\pi}{a} w_{m,n} \sin\frac{n\pi y}{b}$$

$$\left.\left(\frac{\partial w}{\partial y}\right)\right|_{y=0,y=b} = 0 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{n\pi}{b} w_{m,n} \sin\frac{m\pi x}{a}$$

$$\left.\right\} \qquad (33)$$

Equation (33) is equivalent to the set of equations

$$
\left.\begin{aligned}
0 &= w_{1,1} + 3w_{1,3} + 5w_{1,5} + \cdots \\[2ex]
0 &= w_{3,1} + 3w_{3,3} + 5w_{3,5} + \cdots \\[2ex]
0 &= w_{5,1} + 3w_{5,3} + 5w_{5,5} + \cdots
\end{aligned}\right\} \tag{34}
$$

The deflection coefficients $w_{m,n}$ must now be solved from the family of equations (equation (27)) for the linear term in terms of the cubic terms and the pressure coefficients $p_{r,s}$. The expressions for $w_{m,n}$ thus obtained are now substituted into equation (34), and the expressions for pressure coefficients $p_{r,s}$ are obtained from equation (32). The resulting family of equations contains linear terms of pk_r and pk_s and the cubes of the deflection functions $w_{m,n}$.

The method of obtaining the required values of the deflection coefficients $w_{m,n}$ and the edge-moment coefficients pk_r and pk_s consists in assuming values for $\dfrac{w_{1,1}}{h}$ and then solving for $\dfrac{pa^4}{Eh^4}$, $\dfrac{w_{1,3}}{h}$, ..., pk_s, pk_r, ... by successive approximations from the simultaneous equations. The procedure is even more laborious than that for simply supported plates. Two numerical solutions are given, namely solutions of the bending problem for a square plate and for a rectangular plate with length-width ratio of 1.5.

FINITE-DIFFERENCE EQUATIONS OF BOUNDARY-VALUE PROBLEMS

Some fundamental concepts about the finite-difference approximation may be worthy of mention before the partial differential equations are converted into finite-difference expressions.

It is assumed that a function $f(x)$ of the variable x is defined for equidistant values of x. If x is one of the values for which $f(x)$ is defined, $f(x)$ is also defined for the values of $x + k\,\Delta x$, where Δx is the interval between two successive values of x and k is an integer. For the sake of simplicity, the value of the function $y = f(x)$ for $x + k\,\Delta x$ may be written as:

$$
f(x + k\,\Delta x) = y_{x+k\Delta x}
$$

The first difference or the difference of the first order Δy_x of y at the point x is now defined as the increment of the value of y obtained in going from x to $x + \Delta x$:

$$\Delta y_x = y_{x+\Delta x} - y_x$$

It is seen that the increment in the direction of increasing x has been arbitrarily chosen; Δy_x could also be defined by the difference $y_x - y_{x-\Delta x}$. This process is continued and the increment of the first difference obtained in going from x to $x + \Delta x$ is called the difference of second order of y at x; that is,

$$\Delta^2 y_x = \Delta y_{x+\Delta x} - \Delta y_x$$

$$= \left(y_{x+2\Delta x} - y_{x+\Delta x} \right) - \left(y_{x+\Delta x} - y_x \right)$$

$$= y_{x+2\Delta x} - 2y_{x+\Delta x} + y_x$$

In general, the difference of order n is defined by

$$\Delta^n y_x = \Delta^{n-1} y_{x+\Delta x} - \Delta^{n-1} y_x$$

If Δx is chosen equal to unity,

$$y_{x+n\Delta x} = y_{x+n}$$

By the use of this notation, the sequence of differences becomes

$$\Delta y_x = y_{x+1} - y_x$$

$$\Delta^2 y_x = y_{x+2} - 2y_{x+1} + y_x$$

$$\Delta^3 y_x = y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x$$

. . . .

$$\Delta^n y_x = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!(n-r)!} y_{x+n-r} \qquad (35)$$

In many physical problems only differences of even order occur. In such cases it is more convenient to define the differences $\Delta^{2m} y_x$ in the following way:

$$\Delta^2 y_x = y_{x-1} - 2y_x + y_{x+1}$$

That is, $\Delta^2 y_x$ is the increment of the first difference taken on the right- and left-hand sides of the point x. In general,

$$\Delta^{2m} y_x = \Delta^2 \left(\Delta^{2m-2} y_x \right) \qquad (36)$$

In this case a difference of order $2m$ represents a linear expression in $y_{x-m}, y_{x-m+1}, \cdots, y_x, \cdots, y_{x+m-1}, y_{x+m}$.

In replacing partial derivatives by the finite-difference expressions, the differences corresponding to the changes of both the coordinates x and y are considered. With the notations as shown in figure 3, the first differences at a point $A_{m,n}$ in the x- and the y-directions are, respectively:

$$\Delta_x w_{m,n} = w_{m+1,n} - w_{m,n}$$

$$\Delta_y w_{m,n} = w_{m,n+1} - w_{m,n}$$

The three kinds of second differences are as follows:

$$
\Delta_{xx} w_{m,n} = \Delta_x^2 w_{m,n}
$$
$$
= w_{m+1,n} - 2w_{m,n} + w_{m-1,n}
$$
$$
\Delta_{yy} w_{m,n} = \Delta_y^2 w_{m,n}
$$
$$
= w_{m,n+1} - 2w_{m,n} + w_{m,n-1}
$$
$$
\Delta_{xy} w_{m,n} = \Delta_y w_{m+1,n} - \Delta_y w_{m,n}
$$
$$
= \left(w_{m+1,n+1} - w_{m+1,n} \right) - \left(w_{m,n+1} - w_{m,n} \right)
$$
$$
= w_{m+1,n+1} - w_{m+1,n} - w_{m,n+1} + w_{m,n}
$$

(37)

The three kinds of fourth differences, which will be used later, are:

$$
\Delta_{xxxx} w_{m,n} = \Delta_x^4 w_{m,n}
$$
$$
= w_{m+2,n} - 4w_{m+1,n} + 6w_{m,n} - 4w_{m-1,n} + w_{m-2,n}
$$
$$
\Delta_{yyyy} w_{m,n} = \Delta_y^4 w_{m,n}
$$
$$
= w_{m,n+2} - 4w_{m,n+1} + 6w_{m,n} - 4w_{m,n-1} + w_{m,n-2}
$$
$$
\Delta_{xxyy} w_{m,n} = \Delta_{xy}^2 w_{m,n}
$$
$$
= w_{m+1,n+1} - 2w_{m+1,n} + w_{m+1,n-1} - 2w_{m,n+1} + 4w_{m,n}
$$
$$
- w_{m,n-1} + w_{m-1,n+1} - 2w_{m-1,n} + w_{m-1,n-1}
$$

(38)

Partial derivatives may be approximated by finite differences as follows:

$$\frac{\partial w}{\partial x} = \frac{\Delta_x w}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{\Delta_y w}{\Delta y}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\Delta_x^2 w}{\Delta x^2}, \quad \frac{\partial^2 w}{\partial y^2} = \frac{\Delta_y^2 w}{\Delta y^2}$$

$$\frac{\partial^2 w}{\partial x \, \partial y} = \frac{\Delta_{xy} w}{\Delta x \, \Delta y}$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{\Delta_x^4 w}{\Delta x^4}, \quad \frac{\partial^4 w}{\partial y^4} = \frac{\Delta_y^4 w}{\Delta y^4}$$

$$\frac{\partial^4 w}{\partial x^2 \, \partial y^2} = \frac{\Delta_{xy}^2 w}{\Delta x^2 \Delta y^2}$$

$$(39)$$

When these relations are used, the fundamental partial differential equations (1) and (2) may be replaced by the following difference equations:

$$\frac{\Delta_x^4 F}{\Delta x^4} + 2\frac{\Delta_{xy}^2 F}{\Delta x^2 \, \Delta y^2} + \frac{\Delta_y^4 F}{\Delta y^4} = \left(\frac{\Delta_{xy} w}{\Delta x \, \Delta y}\right)^2 - \frac{\Delta_x^2 w}{\Delta x^2} \frac{\Delta_y^2 w}{\Delta y^2}$$

$$\frac{\Delta_x^4 w}{\Delta x^4} + 2\frac{\Delta_{xy}^2 w}{\Delta x^2 \, \Delta y^2} + \frac{\Delta_y^4 w}{\Delta y^4} = 10.8p + 10.8\left(\frac{\Delta_y^2 F}{\Delta y^2} \frac{\Delta_x^2 w}{\Delta x^2}\right.$$

$$\left. + \frac{\Delta_x^2 F}{\Delta x^2} \frac{\Delta_y^2 w}{\Delta y^2} - 2\frac{\Delta_{xy} F}{\Delta x \, \Delta y} \frac{\Delta_{xy} w}{\Delta x \, \Delta y}\right)$$

$$(40)$$

If $\Delta x = \Delta y = \Delta l$, and the relations (37) and (38) are used, equation (40) may be written as

$$
F_{m+2,n} - 8F_{m+1,n} + 20F_{m,n} - 8F_{m-1,n} + F_{m-2,n} + F_{m,n+2} - 8F_{m,n+1}
$$

$$
- 8F_{m,n-1} + F_{m,n-2} + 2F_{m+1,n+1} + 2F_{m+1,n-1} + 2F_{m-1,n+1} + 2F_{m-1,n-1}
$$

$$
= \left(w_{m+1,n+1} - w_{m+1,n} - w_{m,n+1} + w_{m,n} \right)^2 - \left(w_{m+1,n} - 2w_{m,n} + w_{m-1,n} \right)
$$

$$
\times \left(w_{m,n+1} - 2w_{m,n} + w_{m,n-1} \right) \tag{41}
$$

and

$$
w_{m+2,n} - 8w_{m+1,n} + 20w_{m,n} - 8w_{m-1,n} + w_{m-2,n} + w_{m,n+2} - 8w_{m,n+1}
$$

$$
- 8w_{m,n-1} + w_{m,n-2} + 2w_{m+1,n+1} + 2w_{m+1,n-1} + 2w_{m-1,n+1} + 2w_{m-1,n-1}
$$

$$
= 10.8(\Delta l)^4 p + 10.8 \left[\left(F_{m,n+1} - 2F_{m,n} + F_{m,n-1} \right) \right.
$$

$$
\times \left(w_{m+1,n} - 2w_{m,n} + w_{m-1,n} \right) + \left(F_{m+1,n} - 2F_{m,n} + F_{m-1,n} \right)
$$

$$
\times \left(w_{m,n+1} - 2w_{m,n} + w_{m,n-1} \right) - 2\left(F_{m+1,n+1} - F_{m+1,n} - F_{m,n+1} + F_{m,n} \right)
$$

$$
\left. \times \left(w_{m+1,n+1} - w_{m+1,n} - w_{m,n+1} + w_{m,n} \right) \right] \tag{42}
$$

In actually writing these equations for each net point, it is more convenient to employ the finite-difference pattern or so-called relaxation pattern as shown in figure 4 rather than to substitute directly into equations (41) and (42).

In terms of finite differences, the boundary conditions can be formulated in the manner discussed in the following paragraphs.

Simply Supported Edge

The boundary conditions for the simply supported edge $y = 0$ are:

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial x\, \partial y}\right)_{y=0} = 0$$

and, for plates with zero edge compression:

$$\left(\frac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0$$

or, for plates with zero or known edge displacements:

$$\int_x \left[\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2} - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] dv = v$$

Let $n = 0$ denote the edge points along $y = 0$. The finite-difference expressions for the boundary conditions are:

$$w_{m,0} = 0 \qquad \left.\begin{array}{c} \\ \\ \left(\Delta_y^2 w\right)_{m,0} = 0 \end{array}\right\} \tag{43}$$

$$\left.\begin{array}{c} \left(\Delta_{xy}F\right)_{m,0} = 0 \\ \\ \left(\Delta_x^2 F\right)_{m,0} = 0 \end{array}\right\} \tag{44}$$

for plates with zero edge compression and

$$\sum_{n=0}^{k-1} \left[\Delta_x^2 F - \mu \Delta_y^2 F - \frac{1}{2} \left(\Delta_y w \right)^2 \right]_{i,n} = v_i \tag{45}$$

where $n = 0$ and $n = k$ denote points along the two edges $y = 0$ and $y = b$, respectively, and i denotes any point along the line $x = $ Constant in the plate.

Clamped Edge

The boundary conditions for the clamped edge $y = 0$ are:

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial w}{\partial y} \right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} \right)_{y=0} = 0$$

$$\int_x^i \left[\frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2} - \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] dy = v$$

With the same notations as were used for the simply supported edges, the finite-difference expressions are:

$$w_{m,0} = 0$$

$$(\triangle_y w)_{m,0} = 0$$

$$\left(\triangle_y{}^2 F - \mu\triangle_x{}^2 F\right)_{m,0} = 0 \qquad\qquad\qquad (46)$$

$$\sum_{n=0}^{k-1}\left[\triangle_x{}^2 F - \mu\triangle_y{}^2 F - \frac{1}{2}\left(\triangle_y w\right)^2\right]_{i,n} = v_i$$

Riveted Panel with Normal Pressure Greater than That of Surrounding Panels.

The boundary conditions which approximate this case are:

$$(w)_{y=0} = 0$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0$$

$$\left(\frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2}\right)_{y=0} = 0$$

$$\int_x\left[\frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2} - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]dy = 0$$

if $y = 0$ is one of the edges.

Expressed in terms of finite differences, these conditions become:

$$w_{m,0} = 0$$

$$\left(\triangle_y^2 w\right)_{m,0} = 0$$

$$\left(\triangle_y^2 F - \mu\triangle_x^2 F\right)_{m,0} = 0$$

$$\sum_{n=0}^{k-1}\left[\triangle_x^2 F - \mu\triangle_y^2 F - \frac{1}{2}(\triangle_y w)^2\right]_{1,n} = 0$$

(47)

The boundary-value problem which approximates the riveted sheet-stringer panel subjected to uniform normal pressure higher than that of the surrounding panels may be formulated in terms of finite differences.

In order to start with a simpler case, the square flat plate will be discussed, since, on account of symmetry, only one-eighth of the plate need be studied.

The finite-difference approximation of any differential equation requires that every point in the domain to which the equation applies must satisfy the initial differential equation. If the points to be taken are infinite in number, the solution of the difference equations is the exact solution of the corresponding differential equations. But the points to be taken are finite in number, the solution will be approximate, and the degree of approximation will increase as the number of points taken is reduced.

Since the diagonals of a square plate are axes of symmetry, if the boundary conditions along the four sides are the same, $w_{1,k} = w_{k,1}$ and $\epsilon_{1,k} = \epsilon_{k,1}$. The conditions for zero edge displacements may be put into different forms. Since

$$u = \int \frac{\partial u}{\partial x}\, dx = \int \epsilon_x\, dx - \int \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx = 0$$

then

$$\int \epsilon_x\, dx = \int \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx$$

In terms of finite differences,

$$\frac{1}{2}(\epsilon_x)_{0,1} + (\epsilon_x)_{1,1} + \cdots + (\epsilon_x)_{m,1} + \cdots + \frac{1}{2}(\epsilon_x)_{k,1} = \frac{1}{2}(\Delta x)^2 \sum_{m=0}^{k-1} (\Delta_x w)^2_{m,1}$$

$$= \frac{1}{2}(\Delta x)^2 \sum_{m=0}^{k-1} \left(w_{m+1,1} - w_{m,1} \right)^2$$

Similarly, in the y-direction,

$$\frac{1}{2}(\epsilon_y)_{1,0} + (\epsilon_y)_{1,1} + \cdots + (\epsilon_y)_{1,n} + \cdots + \frac{1}{2}(\epsilon_y)_{1,k} = \frac{1}{2}(\Delta y)^2 \sum_{n=0}^{k-1} (\Delta_y w)^2_{1,n}$$

$$= \frac{1}{2}(\Delta y)^2 \sum_{n=0}^{k-1} \left(w_{1,n+1} - w_{1,n} \right)^2$$

The sum of these two equations and the fact that $w_{1,k} = w_{k,1}$, $(\epsilon_x)_{1,k} = (\epsilon_y)_{k,1}$, and $\Delta x = \Delta y = \Delta l$ give

$$\frac{1}{2}(\epsilon_x + \epsilon_y)_{1,0} + (\epsilon_x + \epsilon_y)_{1,1} + \cdots + (\epsilon_x + \epsilon_y)_{1,n} + \cdots + (\epsilon_x + \epsilon_y)_{1,k-1} + \frac{1}{2}(\epsilon_x + \epsilon_y)_{1,k}$$

$$= \frac{1}{(\Delta l)^2} \sum_{n=0}^{k-1} \left(w_{1,n+1} - w_{1,n} \right)^2 \tag{48}$$

Now,

$$\epsilon_x + \epsilon_y = \frac{\partial^2 F}{\partial y^2} - \mu\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} - \mu\frac{\partial^2 F}{\partial y^2}$$

$$= \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)(1 - \mu)$$

$$= \left(\nabla^2 F\right)(1 - \mu)$$

Note that $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Equation (48) then becomes

$$\left(\nabla^2 F\right)_{1,0} + 2\left(\nabla^2 F\right)_{1,1} + \cdots + 2\left(\nabla^2 F\right)_{1,n} + \cdots + 2\left(\nabla^2 F\right)_{1,k-1}$$

$$+ \left(\nabla^2 F\right)_{1,k} = \frac{2}{(1 - \mu)}(\Delta l)^2 \sum_{n=0}^{k-1} \left(w_{1,n+1} - w_{1,n}\right)^2 \qquad (49)$$

This simplification is not necessary, but it is useful in applying the relaxation method.

n = 1.- On referring to figure 5, it is seen that points 1' and 2' are fictitious points placed outside the plate in order to give a better approximation to the boundary conditions.

By the use of $\mu^2 = 0.1$ or $\mu = 0.316228$ for aluminum alloy, the compatibility equation becomes

$$20F_0 - 32F_1 + 8F_2 + 4F_1 = K_0 \qquad (50)$$

where

$$K_0 = (w_2 - 2w_1 + \overset{\bullet}{w}_0)^2 - (2w_1 - 2w_0)^2$$

Then the equilibrium equation is

$$20w_0 - 32w_1 + 8w_2 + 4w_{1'} = p' + 10.8\bigg[2(2F_1 - 2F_0)(2w_1 - 2w_0)$$

$$- 2(w_2 - 2w_1 + w_0)(F_2 - 2F_1 + F_0)\bigg] \qquad (51)$$

where $p' = 12(1 - \mu^2)(\Delta l)^4 p = 0.675p$, since $\Delta l = \frac{1}{2}$.

The boundary conditions are:

(a) $w_1 = 0, \quad w_2 = 0$

(b) $w_{1'} - 2w_1 + w_0 = 0$

(c) $F_0 - 2F_1 + F_{1'} - \mu(2F_2 - 2F_1) = 0$

(d) $(4F_1 - 4F_0) + (F_0 + 2F_2 + F_{1'} - 4F_1) = S_1$

where $S_1 = (w_1 - w_0)^2 \dfrac{2}{(1 - \mu)} = \dfrac{(w_1 - w_0)^2}{0.341886}$. The boundary-value problem now determines the values of w uniquely and the values of F to within an unknown constant. Since the actual value of the constant is irrelevant, it may be defined by letting $F_2 = 0$.

Solving $w_{1'}$, $w_{2'}$ and $F_{1'}$ from the boundary conditions gives the following result:

$$w_{1'} = -w_0$$

$$w_{2'} = -w_1 = 0$$

$$F_{1'} = -F_0 + 2(1 - \mu)F_1$$

When these values are substituted into equations (50), (51), and (d), the resulting equations are

$$16F_0 - 26.529824F_1 = -3w_0^2$$

$$16w_0 = p' + 43.2w_0F_0$$

$$-4F_0 + 1.367544F_1 = \frac{w_0^2}{0.341886}$$

(52)

The eight or nine significant figures in these equations are due to computations made with a computing machine having 10 columns. In order to get satisfactory results in subsequent computations it is convenient to retain a number of figures beyond those normally considered justifiable because of the precision of the basic data.

n = 2.- With reference to figure 6, points 3', 4', and 5' are again fictitious points. The compatibility equations are:

$$20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0$$

$$-8F_0 + 25F_1 - 16F_2 - 8F_3 + 6F_4 + F_{3'} = K_1$$

$$2F_0 - 16F_1 + 22F_2 + 4F_3 - 16F_4 + 2F_5 + 2F_{4'} = K_2$$

(53)

where K_0, K_1, and K_2 are equal to $\left[\left(\Delta_{xy}w\right)^2 - \Delta_x^2w\,\Delta_y^2w\right]$ at points 0, 1, and 2, respectively.

The equilibrium conditions are:

$$20w_0 - 32w_1 + 8w_2 + 4w_3$$

$$= p' + 21.6\left[(\alpha_0' + \beta_0')(w_1 - w_0) - \gamma'(w_0 - 2w_1 + w_2)\right]$$

$$-8w_0 + 25w_1 - 16w_2 - 8w_3 + 6w_4 + w_{3'}$$

$$= p' + 10.8\left[\alpha_1'(2w_2 - 2w_1) + \beta_1'(w_0 - 2w_1 + w_3)\right.$$

$$\left. - 2\gamma_1'(w_4 - w_3 - w_2 + w_1)\right]$$

$$2w_0 - 16w_1 + 22w_2 + 4w_3 - 16w_4 + 2w_5 + 2w_{4'}$$

$$= p' + 10.8\left[\left(\alpha_2' + \beta_2'\right)\left(w_4 - 2w_2 + w_1\right) - 2\gamma_2'\left(w_5 - 2w_4 + w_2\right)\right]$$

$$\left.\right\} \quad (54)$$

where α', β', γ' are $\Delta_x^2 F$, $\Delta_y^2 F$, $\Delta_{xy} F$ at the respective points indicated by the subscripts.

The conditions for zero edge displacements are:

$$-2F_0 - 3F_1 + 4F_2 - 2F_3 + 2F_4 + F_{3'} = S_1$$

$$F_0 - 5F_2 + 2F_3 + F_5 + F_{4'} = S_2$$

$$\left.\right\} \quad (55)$$

where

$$S_1 = \frac{1}{0.341886}\left[(w_1 - w_0)^2 + (w_3 - w_1)^2\right]$$

$$S_2 = \frac{1}{0.341886}\left[(w_2 - w_1)^2 + (w_4 - w_2)^2\right]$$

The boundary conditions are:

(a) $w_3 = 0, \quad w_4 = 0, \quad w_5 = 0$

(b) $w_3' - 2w_3 + w_1 = 0$

$w_4' - 2w_4 + w_2 = 0$

$w_5' - 2w_5 + w_4 = 0$

(c) $F_1 - 2F_3 + F_3' - \mu(2F_4 - 2F_3) = 0$

$F_2 - 2F_4 + F_4' - \mu(F_5 - F_4 + F_3) = 0$

$F_4 - 2F_5 + F_5' = 0$

For the same reason as explained in the case of $n = 1$, let $F_5 = 0$. Solution of the boundary-conditions equations gives

(d) $w_3' = -w_1$

$w_4' = -w_2$

$w_5' = 0$

(e) $F_5' = -F_4$

$F_4' = 2F_4 + \mu(F_3 - 2F_4) - F_2$

$F_3' = 2F_3 + \mu(2F_4 - 2F_3) - F_1$

The combination of the foregoing equations (d) and (e) with equations (53), (54), and (55) gives

$$
\left.\begin{array}{l}
20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0 \\[2mm]
-8F_0 + 24F_1 - 16F_2 - 6.632456F_3 + 6.632456F_4 = K_1 \\[2mm]
2F_0 - 16F_1 + 20F_2 + 4.632456F_3 - 13.264912F_4 = K_2 \\[2mm]
-2F_0 - 4F_1 + 4F_2 - 0.632456F_3 + 2.632456F_4 = S_1 \\[2mm]
F_0 - 6F_2 + 2.316228F_3 + 1.367544F_4 = S_2
\end{array}\right\} \quad (56)
$$

and

$$
\left.\begin{array}{l}
\left[20 + 21.6(\alpha_0' + \beta_0' + \gamma_0')\right] w_0 \\[1mm]
\quad - \left[32 + 21.6(\alpha_0' + \beta_0' + 2\gamma_0')\right] w_1 + (8 + 21.6\gamma_0')w_2 = p' \\[2mm]
-(8 + 10.8\beta_1')w_0 + \left[24 + 21.6(\alpha_1' + \beta_1' + \gamma_1')\right] w_1 \\[1mm]
\quad - \left[16 + 12.6(\alpha_1' + \gamma_1')\right] w_2 = p' \\[2mm]
2w_0 - \left[16 + 10.8(\alpha_2' + \beta_2')\right] w_1 \\[1mm]
\quad + \left[20 + 21.6(\alpha_2' + \beta_2' + \gamma_2')\right] w_2 = p'
\end{array}\right\} \quad (57)
$$

where $p' = 12(1 - \mu^2)(\Delta l)^4 p = 0.0421875p$, since $\Delta l = \frac{1}{4}$.

$\underline{n = 3}$.- Reference is made to figure 7 and to the fact that points 6', 7', 8', and 9' are fictitious points for reasons explained in the case $n = 1$; then the compatibility equations are as follows:

$$20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0$$

$$-8F_0 + 25F_1 - 16F_2 - 8F_3 + 6F_4 + F_6 = K_1$$

$$2F_0 - 16F_1 + 22F_2 + 4F_3 - 16F_4 + 2F_5 + 2F_7 = K_2$$

$$F_0 - 8F_1 + 4F_2 + 20F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_{6'} = K_3$$

$$3F_1 - 8F_2 - 8F_3 + 23F_4 - 8F_5 + 2F_6 - 8F_7 + 3F_8 + F_{7'} = K_4$$

$$2F_2 + 2F_3 - 16F_4 + 20F_5 + 4F_7 - 16F_8 + 2F_9 + 2F_{8'} = K_5$$

$$(58)$$

where K_0, K_1, K_2, K_4, and K_5 are equal to $\left[\left(\Delta_{xy}w\right)^2 - \Delta_x^2 w \Delta_y^2 w\right]$ at points 0, 1, 2, 3, 4, and 5, respectively.

The equilibrium equations are:

$$20w_0 - 32w_1 + 8w_2 + 4w_3 = p' + 21.6\Big[(\alpha_0' + \beta_0')(w_1 - w_0)$$

$$- \gamma_0'(w_0 - 2w_1 + w_2)\Big]$$

$$-8w_0 + 25w_1 - 16w_2 - 8w_3 + 6w_4$$

$$= p' + 10.8\Big[2\alpha_1'(w_2 - w_1) + \beta_1'(w_0 - 2w_1 + w_3)$$

$$- 2\gamma_1'(w_1 - w_2 - w_3 + w_4)\Big]$$

$$2w_0 - 16w_1 + 22w_2 + 4w_3 - 16w_4 + 2w_5$$

$$= p' + 10.8\Big[(\alpha_2' + \beta_2')(w_1 - 2w_2 + w_4) - 2\gamma_2'(w_2 - 2w_4 + w_5)\Big]$$

$$w_0 - 8w_1 + 4w_2 + 20w_3 - 16w_4 + 2w_5 - 8w_6 + 4w_7 + w_6'$$

$$= p' + 10.8\Big[\alpha_3'(2w_4 - 2w_3) + \beta_3'(w_1 - 2w_3 + w_6)$$

$$- 2\gamma_3'(w_3 - w_4 - w_6 + w_7)\Big]$$

$$3w_1 - 8w_2 - 8w_3 + 23w_4 - 8w_5 + 2w_6 - 8w_7 + 3w_8 + w_7'$$

$$= p' + 10.8\Big[\alpha_4'(w_3 - 2w_4 + w_5) + \beta_4'(w_2 - 2w_4 + w_7)$$

$$- 2\gamma_4'(w_4 - w_5 - w_7 + w_8)\Big]$$

$$2w_2 + 2w_3 - 16w_4 + 20w_5 + 4w_7 - 16w_8 + 2w_9 + 2w_8'$$

$$= p' + 10.8\Big[(\alpha_5' + \beta_5')(w_4 - 2w_5 + w_8) - 2\gamma_5'(w_5 - 2w_8 + w_9)\Big]$$

(59)

where α', β', and γ' are $\Delta_x^2 F$, $\Delta_y^2 F$, $\Delta_{xy} F$ at the respective points corresponding to the subscripts, and $p' = 12(1 - \mu^2)(\Delta l)^4 p = 0.00833333p$, since $\Delta l = \frac{1}{6}$.

The conditions for zero edge displacements are:

$$\left.\begin{array}{l} -2F_0 - 2F_1 + 4F_2 - 5F_3 + 4F_4 - 2F_6 + 2F_7 + F_{6'} = S_2 \\[2mm] F_0 - 4F_2 + 3F_3 - 3F_4 + 2F_5 + F_6 - 2F_7 + F_8 + F_{7'} = S_2 \\[2mm] F_1 + 2F_2 - 2F_3 - 2F_4 - 5F_5 + F_6 + 3F_7 + F_9 + F_{8'} = S_3 \end{array}\right\} \qquad (60)$$

where

$$S_1 = \frac{2}{1 - \mu} \sum_{m=0}^{k-1} \left(w_{m+1,i} - w_{m,i}\right)^2$$

The boundary conditions are:

(a) $w_6 = 0$, $w_7 = 0$, $w_8 = 0$, $w_9 = 0$

(b) $w_{6'} - 2w_6 + w_3 = 0$

$w_{7'} - 2w_7 + w_4 = 0$

$w_{8'} - 2w_8 + w_5 = 0$

$w_{9'} - 2w_9 + w_8 = 0$

(c) $F_3 - 2F_6 + F_{6'} - \mu(2F_7 - 2F_6) = 0$

$F_4 - 2F_7 + F_{7'} - \mu(F_8 + F_6 - 2F_7) = 0$

$F_5 - 2F_8 + F_{8'} - \mu(F_9 + F_7 - 2F_8) = 0$

$F_{9'} - 2F_9 + F_8 = 0$

Solutions of the boundary-conditions equations give

(d) $w_{6'} = -w_3$

$w_{7'} = -w_4$

$w_{8'} = -w_5$

$w_{9'} = 0$

(e) $F_{6'} = -F_3 + 1.367544F_6 + 0.632456F_7$

$F_{7'} = -F_4 + 1.367544F_7 + 0.316228F_6 + 0.316228F_8$

$F_{8'} = -F_5 + 1.367544F_8 + 0.316228F_7$

$F_{9'} = -F_8$

where $F_9 = 0$ is assumed for the same reason as explained in the case of $n = 1$.

Combination of the foregoing equations gives:

$$20F_0 - 32F_1 + 8F_2 + 4F_3 = K_0$$

$$-8F_0 + 25F_1 - 16F_2 - 8F_3 + 6F_4 + F_6 = K_1$$

$$2F_0 - 16F_1 + 22F_2 + 4F_3 - 16F_4 + 2F_5 + 2F_7 = K_2$$

$$F_0 - 8F_1 + 4F_2 + 18F_3 - 16F_4 + 2F_5 - 6.632456F_6 + 4.632456F_7 = K_3$$

$$3F_1 - 8F_2 - 8F_3 + 22F_4 - 8F_5 + 2.316228F_6 - 6.632456F_7$$
$$+ 3.316228F_8 = K_4$$

$$2F_2 + 2F_3 - 16F_4 + 18F_5 + 4.632456F_7 - 13.264912F_8 = K_5$$

$$-2F_0 - 2F_1 + 4F_2 - 6F_3 + 4F_4 - 0.632456F_6 + 2.632456F_7 = S_1$$

$$F_0 - 4F_2 + 3F_3 - 4F_4 + 2F_5 + 1.316228F_6 - 0.632456F_7$$
$$+ 1.316228F_8 = S_2$$

$$F_1 + 2F_2 - 2F_3 - 2F_4 - 6F_5 + F_6 + 3.316228F_7 + 1.367544F_8 = S_3$$

(61)

and

$$\left[20 + 21.6(\alpha_0' + \beta_0' + \gamma_0')\right]w_0 - \left[32 + 21.6(\alpha_0' + \beta_0' + \gamma_0')\right]w_1$$

$$+ \left(8 + 21.6\gamma_0'\right)w_2 + 4w_3 = p'$$

$$- \left(8 + 10.8\beta_1'\right)w_0 + \left[25 + 21.6(\alpha_1' + \beta_1' + \gamma_1')\right]w_1$$

$$- \left[16 + 21.6(\alpha_1' + \gamma_1')\right]w_2 - \left[8 + 10.8(\beta_1' + 2\gamma_1')\right]w_3$$

$$+ (6 + 21.6\gamma_1')w_4 = p'$$

$$2w_0 - \left[16 + 10.8(\alpha_2' + \beta_2')\right]w_1 + \left[22 + 21.6(\alpha_2' + \beta_2' + \gamma_2')\right]w_2$$

$$+ 4w_3 - \left[16 + 10.8(\alpha_2' + \beta_2' + 4\gamma_2')\right]w_4$$

$$+ (2 + 21.6\gamma_2')w_5 = p'$$

$$w_0 - \left(8 + 10.8\beta_3'\right)w_1 + 4w_2 + \left[19 + 21.6(\alpha_3' + \beta_3' + \gamma_3')\right]w_3$$

$$- \left[16 + 21.6(\alpha_3' + \gamma_3')\right]w_4 + 2w_5 = p'$$

$$3w_1 - (8 + 10.8\beta_4')w_2 - (8 + 10.8\alpha_4')w_3$$

$$+ \left[22 + 21.6(\alpha_4' + \beta_4' + \gamma_4')\right]w_4 - \left[8 + 10.8(\alpha_4' + 2\gamma_4')\right]w_5 = p'$$

$$2w_2 + 2w_3 - \left[16 + 10.8(\alpha_5' + \beta_5')\right]w_4$$

$$+ \left[18 + 21.6(\alpha_5' + \beta_5' + \gamma_5')\right]w_5 = p'$$

(62)

METHOD OF SUCCESSIVE APPROXIMATIONS

Explanation

After the boundary-value problems are expressed in terms of finite-difference equations, two sets of simultaneous equations are obtained. The first set consists of the compatibility equations and the equations specifying the condition of zero edge displacements. These equations contain linear terms of the nondimensional stress function F and the second-order terms of the nondimensional deflection w, and are of the form

$$c_{00}F_0 + c_{01}F_1 + \cdots + c_{0n}F_n = K_0$$

$$c_{10}F_0 + c_{11}F_1 + \cdots + c_{1n}F_n = K_1$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

and

$$c'_{10}F_0 + c'_{11}F_1 + \cdots + c'_{1n}F_n = S_1$$

$$(63)$$

where $K_1 = \left(\Delta_{xy}w\right)^2 - \left(\Delta_x^2 w\right)\left(\Delta_y^2 w\right)$ at points 0, 1, and so forth, corresponding to the subscripts of K; $S_1 = \dfrac{2}{1 - \mu} \sum_m \left(\Delta_x w\right)^2_{m,1}$; and c_{00}, c_{01}, $\cdots$, c'_{10}, c'_{11}, $\cdots$ are given constants.

The second set consists of the equilibrium equations, which contain the linear terms of w with coefficients involving linear terms in F and are of the form

$$\left(a_{00} + b_{00}\alpha'_0 + b'_{00}\beta'_0 + b''_{00}\gamma'_0\right)w_0$$

$$+ \left(a_{01} + b_{01}\alpha'_0 + b'_{01}\beta'_0 + b''_{01}\gamma'_0\right)w_1$$

$$+ \cdots + \left(a_{0n} + b_{0n}\alpha'_0 + b'_{0n}\beta'_0 + b''_{0n}\gamma'_0\right)w_n = p'$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (64)$$

where $\alpha' = \triangle_x{}^2F$, $\beta' = \triangle_y{}^2F$, $\gamma' = \triangle_{xy}F$ at points $0, 1, \ldots$ corresponding to the subscripts of α', β', γ', and a_{00}, a_{01}, $\ldots$, b_{00}, b_{01}, $\ldots$, b'_{00}, b'_{01}, $\ldots$, b''_{00}, b''_{01}, $\ldots$ are given constants.

If a set of values of w is assumed at each of the net points and the values of K_i and S_i are computed, equation (63) becomes a system of linear simultaneous equations in F and can therefore be solved exactly by Crout's method for solving systems of linear simultaneous equations (reference 24). After the values of F have been computed from equation (63), values of α', β', and γ' can be found without any difficulty. Then equation (64) becomes another system of linear simultaneous equations and may be solved exactly by Crout's method again. If the values of w found from equation (64) check with those assumed, the problem is completely solved.

In most cases, however, the values of w will not check with each other. By following the usual method of successive approximations, the computed w's will now replace the assumed ones and the cycle of computations will be repeated. If the value of w at the end of the cycle still does not check with the one assumed at the beginning of the cycle, another cycle will be performed. In this problem, however, if the ordinary method were followed, the results would be found to diverge, oscillating to infinity. Therefore, a special procedure must be devised to make the process converge.

A simple case will be examined first. In the boundary-value problem in which $n = 1$ under the normal pressure $p = 100$, equation (52) can easily be reduced to the form

$$w_0 = \frac{p'}{16 + 37.6908w_0{}^2}$$

or

$$w_0{}^3 + 0.424507w_0 - 1.790888 = 0 \qquad (65)$$

The third-order algebraic equation can easily be solved, and the roots of this equation are

$$w_0 = 1.098254 \quad \text{and} \quad (-0.549127 \pm 1.1528781)$$

For the physical problem, only the real root is of interest because the imaginary roots do not have any physical meaning.

An attempt will now be made to solve equation (65) by the usual method of successive approximations. It is assumed that

$$w_0 = 1.200000, \quad w_0{}^2 = 1.440000$$

$$w_0 = \frac{67.5}{70.27474} = 0.960516$$

$$w_0{}^2 = 0.922591$$

If it is assumed that $w_0{}^2 = 0.922591$ for the second cycle and that the value of $w_0{}^2$ found from the second cycle is the value for the third cycle, and so on, the following values of $w_0{}^2$ are found from various cycles:

$$1.767416, \quad 0.667554, \quad 2.689324, \quad \text{and so forth.}$$

These values are oscillatorily divergent. A plot of these values against cycles shows that they oscillate about the true value 1.206161, and the true value is approximately the mean of the values obtained from two consecutive cycles (fig. 8).

If $w_0{}^2 = \frac{1}{2}(1.440000 + 0.922591) = 1.181296$ is taken as the assumed value of $w_0{}^2$ for the second cycle, and the mean of this value and the value found from the second cycle are taken as the assumed value for the third cycle, and so forth, the values of $w_0{}^2$ are found from various cycles as follows:

Cycles	2	3	4	5
w_0^2 assumed	1.181296	1.212550	1.204658	1.206524
w_0^2 found	1.243805	1.196766	1.208390	1.204526

Cycles	6	7
w_0^2 assumed	1.206075	1.206182
w_0^2 found	1.206289	1.206131

This process is convergent and w_0 converges to the real root of equation (65). The value of w_0 found at the end of the seventh cycle is 1.098240 and is accurate to four figures at the end of the fifth cycle, in which case it is found to be 1.098010. The results are plotted against cycles in figure 9.

It is to be noted that $K_0 = -3w_0^2$ in the case of $n = 1$. The values obtained by the method of successive approximations would converge if K_0 were assumed to be the mean value of two consecutive cycles. It is found that this convergent property is the same for $n > 1$. If the mean of K's or S's found from two consecutive cycles is taken, the values are convergent but are oscillatorily divergent if the usual way of successive approximations is followed.

It may be pointed out here that for the special case $n = 1$, if the mean of the values of w_0 from two consecutive cycles is used, the values are also convergent, and if w_0^2 for the second cycle is assumed to be equal to the sum of 0.6 times the assumed value for the first cycle and 0.4 times the value found from the first cycle, and so on, the convergence is much more rapid (fig. 10), but this result is not true for the cases with $n > 1$.

The rapidity of the convergence depends on the accuracy of the assumed values of K's and S's for the first trial. The deflection w from the linear small-deflection theory can easily be determined. When p is small, the values of w so determined would give a first approximation to the problem. It is convenient, therefore, to start the computation when p is small and then to consider the cases when p is large. Also it is advisable to begin with but a few net points and then gradually to increase the number of net points. For example, consider case $n = 1$. When w_0 is found for a certain small p, a curve of w_0 against p can be plotted because the slope of the curve at the origin can be determined from the small-deflection theory. For a larger value of p, w_0 can now be estimated by extrapolation. For $n = 2$, the value of w_0 found for $n = 1$ can be used as a first trial. However, w_2 and w_3 are still difficult to estimate. In order to obtain first approximations to these quantities, the ratios w_2/w_0 and w_3/w_0 may be found from the small-deflection theory and the values

of w_2 and w_3 computed by multiplying these ratios by the estimated value of w_0. When the deflections have been assumed at every point of the net, the values of K and S can be computed. These are the values which may be used as a first trial. By successive approximations, the true values of the w's are then determined. The values of w_0 and the (w_n/w_0)'s are now plotted against p to estimate the corresponding values at a larger p. The values estimated by extrapolation may be used as the trial values corresponding to that p. The process is repeated until the maximum p is reached. For $n = 3$, w_0 from $n = 2$ is used as a first trial; the remainder of the procedure is the same as before.

Sample Calculations

Finite-difference solutions of small-deflection theory.- The small-deflection theory of the simply supported square plate will be studied first. The differential equation is

$$\nabla^4 w = \frac{p}{D} \qquad (66)$$

and the boundary conditions are

$$
\left.
\begin{aligned}
w &= 0 \quad \text{along four edges} \\[2mm]
\frac{\partial^2 w}{\partial x^2} &= 0 \quad \text{along} \quad x = \pm\frac{a}{2} \\[2mm]
\frac{\partial^2 w}{\partial y^2} &= 0 \quad \text{along} \quad y = \pm\frac{a}{2}
\end{aligned}
\right\} \qquad (67)
$$

where a is the length of the sides.

With equations (66) and (67) written nondimensionally by letting $w' = \frac{w}{h}$, $p' = \frac{pa^4}{Eh^4}$, $x' = \frac{x}{a}$, and $y' = \frac{y}{b}$, where w', p', and x' and y' are nondimensional deflection, pressure, and lengths, respectively, and with the primes dropped, the boundary-value problem is:

$$\nabla^4 w = 12(1 - \mu^2)p$$

$$w = 0 \quad \text{at} \quad x = \pm\frac{1}{2}, \quad y = \pm\frac{1}{2}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = \pm\frac{1}{2}$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = \pm\frac{1}{2}$$

By retaining the notations previously used, the finite-difference equations for the problem are

$$\left.\begin{array}{c} \Delta_x^4 w + 2\Delta_{xy}^2 w + \Delta_y^4 w = p' \\[2ex] (w)_{x=\pm\frac{1}{2}, y=\pm\frac{1}{2}} = 0 \\[2ex] \left(\Delta_x^2 w\right)_{x=\pm\frac{1}{2}} = 0 \\[2ex] \left(\Delta_y^2 w\right)_{y=\pm\frac{1}{2}} = 0 \end{array}\right\} \qquad (68)$$

where $p' = 12(1 - \mu^2)(\Delta l)^4 p$.

For $n = 1$ (fig. 5), the finite-difference equation, after the boundary conditions are employed, becomes

$$16w_0 = p'$$

therefore,

$$w_0 = 0.0625p'$$
$$= 0.042188p$$

for $\mu^2 = 0.1$.

For n = 2 (fig. 6), the finite-difference equations, after the boundary conditions are inserted, become

$$\left.\begin{array}{l} 20w_0 - 32w_1 + 8w_2 = p' \\[2ex] -8w_0 + 24w_1 - 16w_2 = p' \\[2ex] 2w_0 - 16w_1 + 20w_2 = p' \end{array}\right\} \qquad (69)$$

When Crout's method is used to solve these equations the solutions of equation (69) are

$$w_0 = 1.031250p' = 0.043506p$$

$$w_1 = 0.750000p' = 0.031641p$$

$$w_2 = 0.546875p' = 0.023071p$$

where μ^2 is taken to be equal to 0.1. For $\mu = 0.3$,

$$w_0 = 0.032989p$$

For n = 3 (fig. 7), the finite-difference equations, after the boundary conditions are employed, become

$$\left.\begin{array}{l} 20w_0 - 32w_1 + 8w_2 + 4w_3 = p' \\[1.5ex] -8w_0 + 25w_1 - 16w_2 - 8w_3 + 6w_4 = p' \\[1.5ex] 2w_0 - 16w_1 + 22w_2 + 4w_3 - 16w_4 + 2w_5 = p' \\[1.5ex] w_0 - 8w_1 + 4w_2 + 19w_3 - 16w_4 + 2w_5 = p' \\[1.5ex] 3w_1 - 8w_2 - 8w_3 + 22w_4 - 8w_5 = p' \\[1.5ex] 2w_2 + 2w_3 - 16w_4 + 18w_5 = p' \end{array}\right\} \qquad (70)$$

The solutions of equation (70) are:

$$w_0 = 5.246672p' = 0.043722p$$

$$w_1 = 4.597633p' = 0.038314p$$

$$w_2 = 4.031250p' = 0.033594p$$

$$w_3 = 2.735207p' = 0.022793p$$

$$w_4 = 2.402367p' = 0.020020p$$

$$w_5 = 1.439164p' = 0.011993p$$

if μ^2 is assumed to be 0.1. If μ is assumed to be 0.3, the answer is

$$w_0 = 0.044208p$$

Timoshenko gives the exact value of w_0 for a simply supported square plate (reference 27) as:

$$w_0 = 0.0443p$$

Therefore the solution by finite differences with $n = 3$ is in error by 0.23 percent. This solution is seen to be sufficiently accurate for engineering purposes. The agreement of the finite-difference approximation with the more exact results of Timoshenko is sufficiently close to encourage application of the finite-difference approximation to the problems with large deflections.

The large-deflections problem, $n = 2$.- After the boundary conditions are inserted, the two sets of finite-difference equations are:

$$
\left.
\begin{aligned}
20F_0 - 32F_1 + 8F_2 + 4F_3 &= K_0 \\
-8F_0 + 24F_1 - 16F_2 - 6.632456F_3 + 6.632456F_4 &= K_1 \\
2F_0 - 16F_1 + 20F_2 + 4.632456F_3 - 13.264912F_4 &= K_2 \\
-2F_0 - 4F_1 + 4F_2 - 0.632456F_3 + 2.632456F_4 &= S_1 \\
F_0 - 6F_2 + 2.316228F_3 + 1.367544F_4 &= S_2
\end{aligned}
\right\} \quad (71)
$$

and

$$\left[20 + 21.6(\alpha_0' + \beta_0' + \gamma_0')\right]w_0 - \left[32 + 21.6(\alpha_0' + \beta_0' + 2\gamma_0')\right]w_1$$

$$+ (8 + 21.6\gamma_0')w_2 = p'$$

$$-(8 + 10.8\beta_1')w_0 + \left[24 + 21.6(\alpha_1' + \beta_1' + \gamma_1')\right]w_1 \qquad\qquad (72)$$

$$- \left[16 + 21.6(\alpha_1' + \gamma_1')\right]w_2 = p'$$

$$2w_0 - \left[16 + 10.8(\alpha_2' + \beta_2')\right]w_1 + \left[20 + 21.6(\alpha_2' + \beta_2' + \gamma_2')\right]w_2 = p'$$

It is to be noted that the terms of the left-hand side of equation (71
do not change if the assumed values of K and S are changed.
Equation (71) can be solved uniquely, therefore, in terms of K's
and S's. The given, auxiliary, and final matrices obtained by Crout's
method are given in tables 1, 2, and 3, respectively. More significant
figures than required are used to ensure good results.

The solutions of equation (71) are as follows:

$$F_0 = -0.048703K_0 - 0.265696K_1 - 0.225111K_2 - 0.304114S_1 - 0.309525S_2$$

$$F_1 = -0.111203K_0 - 0.307363K_1 - 0.235527K_2 - 0.262447S_1 - 0.288692S_2$$

$$F_2 = -0.103085K_0 - 0.311962K_1 - 0.221052K_2 - 0.162880S_1 - 0.317642S_2 \quad (73)$$

$$F_3 = -0.189937K_0 - 0.506498K_1 - 0.316561K_2 - 0.253249S_1 - 0.126624S_2$$

$$F_4 = -0.094968K_0 - 0.316561K_1 - 0.269077K_2 - 0.063312S_1 - 0.221593S_2$$

For a numerical example of the computation, let

$$p = 100$$

$$p' = 0.0421875p = 4.218750$$

From the curves for $w_0 \sim p$, $\dfrac{w_1}{w_0} \sim p$, and $\dfrac{w_2}{w_0} \sim p$ (figs. 11 and 12), it is estimated that

$$w_0 = 1.135$$

$$\frac{w_1}{w_0} = 0.7535$$

$$\frac{w_2}{w_0} = 0.5775$$

The first trial values are

$$w_0 = 1.135$$

$$w_1 = 0.855222$$

$$w_2 = 0.655463$$

These values are written at the right-hand corners below the corresponding net points. The finite-difference patterns are used as given in figure 4, and α, β, γ, $w_{n+1} - w_n$, and then K and S are found at the net points (fig. 13). As an example,

$$\alpha_0 = \beta_0 = -2(1.135000 - 0.855222) = -0.559556$$

$$\gamma_0 = 1.135000 + 0.655463 - 2 \times 0.855222 = 0.080019$$

$$K_0 = (0.080019)^2 - (-0.559556)^2 = -0.306700$$

Similarly, it is found that

$$K_1 = -0.189997$$

$$K_2 = 0.221966$$

$$S_1 = 2.368276$$

$$S_2 = 1.373368$$

From equation (73) the values of F's are obtained as follows:

$$F_0 = -1.129866$$

$$F_1 = -0.977802$$

$$F_2 = -0.780162$$

$$F_3 = -0.689444$$

$$F_4 = -0.424723$$

These values are substituted in any one of the expressions (equation (71)) as a check and then are recorded at the net points, as in figure 13. Similarly, the values of α', β', and γ' are recorded below the corresponding values of F.

Equation (72) can now be written and the given matrix is

w_0	w_1	w_2	= p'	Check column
34.122771	-47.107213	8.984442	4.218750	0.218750
-12.269024	36.930948	-20.392900	4.218750	8.487774
2.000000	-19.408458	28.313451	4.218750	15.123743

The check column can be obtained by using the following relation:

Check column

$$-4 + p'$$

$$10.8\beta_1' + p'$$

$$6 + 21.6(\alpha_2' + \beta_2') + p'$$

The sum of the elements in a row should be equal to the value of the element of the same row in the check column. This procedure provides a check for the substitution made in the given matrix.

The first approximation gives, therefore

$$w_0 = 1.117078$$

$$w_1 = 0.843225$$

$$w_2 = 0.648112$$

A computation similar to the one outlined in the foregoing numerical example gives

$$K_0 = -0.293781$$

$$K_1 = -0.184115$$

$$K_2 = 0.214841$$

$$S_1 = 2.299072$$

$$S_2 = 1.339974$$

As a second trial, assume

$$K_0 = \frac{1}{2}(-0.306700 - 0.293781) = -0.300241$$

$$K_1 = \frac{1}{2}(-0.189997 - 0.184115) = -0.187056$$

$$K_2 = \frac{1}{2}(0.221966 + 0.214841) = 0.218404$$

$$S_1 = \frac{1}{2}(2.368276 + 2.299072) = 2.333673$$

$$S_2 = \frac{1}{2}(1.373368 + 1.339974) = 1.356671$$

The results of the second, third, and fourth trials are shown in figure 13. The corresponding assumed and computed values of the fourth trial are

	Assumed	Computed
K_0	-0.300446	-0.300006
K_1	-.187407	-.187472
K_2	.218738	.218786
S_1	2.337941	2.338531
S_2	1.360090	1.360631

The first three values check with one another, and the results, corrected
to the third decimal place, are

$$w_0 = 1.1269$$

$$w_1 = 0.8502$$

$$w_2 = 0.6528$$

The large-deflections problem, n = 3.- When n is taken to be
greater than 2, the same procedure of computation as that in the case
of n = 2 is still valid. As an example, the case of n = 3 will be
considered, when the square plate is subjected to a uniform pressure
of p = 100.

After using the boundary conditions, the two sets of difference
equations (61) and (62) are obtained. Equation (61) can be solved in
terms of K's and S's, and the results are given in table 4.

From the curves of $w_0 \sim p$, $\dfrac{w_1}{w_0} \sim p$, $\dfrac{w_2}{w_0} \sim p$, $\dfrac{w_3}{w_0} \sim p$, $\dfrac{w_4}{w_0} \sim p$,
and $\dfrac{w_5}{w_0} \sim p$ (figs. 12 and 14), the following values are obtained by
extrapolation:

$$w_0 = 1.1247$$

$$\frac{w_1}{w_0} = 0.8891$$

$$\frac{w_2}{w_0} = 0.7932$$

$$\frac{w_3}{w_0} = 0.5516$$

$$\frac{w_4}{w_0} = 0.5037$$

$$\frac{w_5}{w_0} = 0.3497$$

For a first trial, it is assumed that

$$w_0 = 1.124700$$

$$w_1 = 0.999971$$

$$w_2 = 0.892112$$

$$w_3 = 0.620385$$

$$w_4 = 0.566511$$

$$w_5 = 0.393308$$

Again these values are written at the right-hand corners below the corresponding net points. With the computed values of α, β, γ, $\Delta_x w$, and $\Delta_y w$, the following values are obtained:

$$K_0 = -0.061945$$

$$K_1 = -0.052063$$

$$K_2 = -0.024186$$

$$K_3 = -0.023043$$

$$K_4 = 0.001252$$

$$K_5 = 0.106245$$

$$S_1 = 1.592696$$

$$S_2 = 1.282838$$

$$S_3 = 0.548700$$

By table 4 the values of F's are found to be

$$F_0 = -1.095495$$

$$F_1 = -1.028996$$

$$F_2 = -0.950911$$

$$F_3 = -0.868159$$

$$F_4 = -0.762520$$

$$F_5 = -0.505761$$

$$F_6 = -0.675850$$

$$F_7 = -0.546620$$

$$F_8 = -0.239729$$

The values of F's are written at the left-hand corners below the corresponding net points, and the values of α', β', and γ' are computed.

When the values of α', β', and γ' are substituted into equation (62) and it is noted that $p' = 0.00833333p = 0.833333$, the given matrix of the equations is obtained as in table 5 and the auxiliary matrix as in table 6, and the solutions of equation (62) given by the final matrix are

$$w_0 = 1.123384$$

$$w_1 = 0.998956$$

$$w_2 = 0.891465$$

$$w_3 = 0.620342$$

$$w_4 = 0.565591$$

$$w_5 = 0.390999$$

It might be pointed out here that the check column of the given matrix may be obtained by a direct substitution by using the following relations:

Check column

$$p'$$

$$-1 + p'$$

$$-2 + p'$$

$$2 \times 10.8\beta_4' + p'$$

$$1 \times 10.8\beta_5' + p'$$

$$6 + 21.6(\alpha_6' + \gamma_6') + p'$$

This procedure would provide a way of checking the substitution in the given matrix, since the sum of the elements in any row should be equal to the element of the same row in the check column.

The values of K_0, K_1, K_2, K_3, K_4, K_5, S_1, S_2, and S_3 are found from the computed values of w's. The mean values of the K's and S's first assumed and those computed are used as the trial values for the second cycle, and so on. At the end of the third trial, the following assumed and computed values are obtained:

	Assumed	Computed
K_0	-0.061763	-0.061695
K_1	-.051947	-.051894
K_2	-.024660	-.024799
K_3	-.023377	-.023477
K_4	.001614	.001697
K_5	.106177	.106204
S_1	1.592106	1.592078
S_2	1.281878	1.281814
S_3	.546560	.546173

These values check with one another to the fourth decimal place. The
deflections at the various net points, accurate to the fourth decimal
place, are

$$w_0 = 1.1240$$

$$w_1 = 0.9995$$

$$w_2 = 0.8920$$

$$w_3 = 0.6207$$

$$w_4 = 0.5660$$

$$w_5 = 0.3915$$

The results of various trials are shown in figure 15.

RELAXATION METHOD

When a more accurate result is needed, the plate must be divided
into a set of finer nets. The number of simultaneous equations
increases as the number of nets is increased. In order to avoid the
solution of simultaneous equations, Southwell's relaxation method
may be used. The so-called relaxation method is essentially a clever
scheme for guessing the solution of a system of difference equations.
A brief description of the method and a numerical example, the small-
deflection problem of a square plate, are given in appendix A.

The solution of the general case of the large-deflection problems
of rectangular plates by the relaxation method has been studied by Green
and Southwell and their method was outlined previously. Green and
Southwell worked with the three complicated equilibrium equations in
terms of the displacements u and v and the deflection w. However,
it is satisfactory to use the two much simpler equations in terms of the
stress function F and the deflection w.

The fundamental differential equations (1) and (2) can be rewritten
as follows:

$$\nabla^4 F = k \tag{74}$$

$$\nabla^4 w = 10.8p + 10.8k' \tag{75}$$

where

$$k = \left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$$

$$k' = \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \, \partial y} \frac{\partial^2 w}{\partial x \, \partial y}$$

In applying the relaxation method, as usual, the domain of the problem to be solved is first drawn, and the net points chosen. Since there are two simultaneous equations to be solved, two sheets of paper may be used, one for F and one for w. A set of solutions of F and w are guessed and are recorded on the F- and w-planes, respectively.

By starting from the assumed values of w, K can be computed without difficulty. Equation (74) is then a linear differential equation for F, and the biharmonic relaxation pattern may be used. After the residues at each point have been reduced to the desired extent, the new values of F may be substituted into equation (75) and it may be solved by the relaxation method. Equation (75) leads to a rather complicated relaxation pattern for w. In actual computations the biharmonic pattern may be used, the assumed values of w being used for the computation of k'. By means of the relaxation process, the residues at all points are reduced somewhat. New values of k' are computed and the residues are then corrected. The relaxation operation is applied again until the values of w are determined to the desired accuracy. The average values of the new K's and S's and the originally assumed ones are now used in the second cycle. The cycles are now repeated until the final results have the desired accuracy.

In general, the boundary conditions for F are usually difficult to handle. It is possible, however, to solve the boundary values of F in terms of its values for interior points. The boundary values of F vary from time to time as the interior values change. The operation is rather complicated, but it can be handled.

In the case of a square plate with given edge displacements, the boundary conditions as given by equation (49) may be used to some advantage. Equation (74) can be written as

$$\left. \begin{aligned} \nabla^2 T &= k \\[2em] \nabla^2 F &= T \end{aligned} \right\} \tag{76}$$

and the boundary conditions are given by

$$
\left.
\begin{aligned}
T_{0,i} + 2T_{1,i} + \cdots + 2T_{m-1,i} + T_{m,i} &= S_i \\
&= \frac{2}{(1-\mu)(\Delta l)^2} \sum_{n=0}^{m} \left(w_{n,i} - w_{n-1,i}\right)^2
\end{aligned}
\right\}
$$

and

$$
\left(\Delta_x^{\,2}F - \mu\Delta_y^{\,2}F\right)_{m,i} = 0
$$

$$(77)$$

In using this form, not only are the boundary conditions much easier to handle, but also the relaxation pattern is simplified from the biharmonic type to the harmonic type. The simplification is obtained at the expense of introducing one more equation into the system and therefore considering one more plane. The results obtained are given in figure 16.

DISCUSSION OF RESULTS

The bending problem of a square plate under uniform normal pressure, with the edges prevented from displacements along the supports but free to rotate about them, is studied by the finite-difference approximations. The difference equations are solved by the method of successive approximation and by the relaxation method. The computation starts with $n = 1$ to $n = 3$, in which case the plate is divided into 36 square nets with 25 inner points. The maximum normal pressure calculated is $\dfrac{pa^4}{Eh^4} = 250$.

After the values of w and F have been determined, the stresses can be found by the following relations:

$$
\sigma_x' = \frac{\partial^2 F}{\partial y^2} = \frac{\Delta_y^{\,2}F}{(\Delta l)^2} = \frac{\beta'}{(\Delta l)^2}
$$

$$
\sigma_y' = \frac{\partial^2 F}{\partial x^2} = \frac{\Delta_x^{\,2}F}{(\Delta l)^2} = \frac{\alpha'}{(\Delta l)^2}
$$

$$\sigma_x{}'' = -\frac{1}{2(1 - \mu^2)}\left(\Delta_x{}^2 w + \mu\Delta_y{}^2 w\right)\frac{1}{(\Delta l)^2}$$

$$= -\frac{1}{2(1 - \mu^2)(\Delta l)^2}(\alpha + \mu\beta)$$

$$\sigma_y{}'' = -\frac{1}{2(1 - \mu^2)(\Delta l)^2}(\beta + \mu\alpha)$$

where σ' and σ'' are the membrane stress and the extreme-fiber bending stress, respectively. The total stresses σ are the sum of the membrane and bending stresses at the section and are maximum at the extreme fiber of the plate. They are

$$\sigma_x = \sigma_x{}' + \sigma_x{}''$$

$$\sigma_y = \sigma_y{}' + \sigma_y{}''$$

At the center of the square plate, $\alpha' = \beta'$ and $\alpha = \beta$, and therefore the stresses are

$$\sigma_x{}' = \sigma_y{}' = \frac{\alpha'}{(\Delta l)^2} = \frac{\beta'}{(\Delta l)^4}$$

$$\sigma_x{}'' = \sigma_y{}'' = \frac{\alpha}{2(1 - \mu)(\Delta l)^2} = \frac{\beta}{2(1 - \mu)(\Delta l)^2}$$

The deflections at various points determined in the cases $n = 1$, $n = 2$, and $n = 3$ are tabulated in tables 7 to 9. The center deflections are plotted against the normal pressure ratio in figure 12. The membrane stresses in the center of the plate and at the centers of the edges are tabulated in table 10 and are plotted in figure 17. The bending and total stresses are tabulated in table 11 and are plotted in figure 18.

A study of the results shows that the maximum error in center deflections is 0.47 percent for $n = 2$ in comparison with $n = 3$ and the maximum error in the center membrane stresses is 0.44 percent, both values being conservative. Both maximum errors occurred at $\frac{pa^4}{Eh^4} = 250$.

The error in the center bending stresses is 2 percent at $\frac{pa^4}{Eh^4} = 12.5$ and is 0.83 percent at $\frac{pa^4}{Eh^4} = 250$, both values being unsafe. The error in the center extreme-fiber stresses is 1.6 percent at $\frac{pa^4}{Eh^4} = 12.5$ and 0.17 percent at $\frac{pa^4}{Eh^4} = 250$, both values being safe. The error in the membrane stresses at the center of the sides is 12 percent for both $\sigma_x'a^2/Eh^2$ and $\sigma_y'a^2/Eh^2$ at $\frac{pa^4}{Eh^4} = 12.5$ and 8.9 percent for both $\sigma_x'a^2/Eh^2$ and $\sigma_y'a^2/Eh^2$ at $\frac{pa^4}{Eh^4} = 250$, these values being unsafe.

One case of $n = 4$ has been solved by the relaxation method. At $\frac{pa^4}{Eh^4} = 100$, it is found that $\frac{w_0}{h} = 1.1250$, $\frac{\sigma_{x_0}'a^2}{Eh^2} = \frac{\sigma_{y_0}'a^2}{Eh^2} = 4.786$, $\frac{\sigma_0 a^2}{Eh^2} = 11.394$, $\frac{\sigma_{x_1}'a^2}{Eh^2} = 9.688$, and $\frac{\sigma_{y_1}'a^2}{Eh^2} = 3.064$. When the results for $n = 3$ are compared with those for $n = 4$ at $\frac{pa^4}{Eh^4} = 100$, the center deflection has an error of 0.09 percent, the center membrane stress has an error of 0.02 percent, the center total stress has an error of 0.5 percent, and the membrane stresses $\sigma_x'a^2/Eh^2$ and $\sigma_y'a^2/Eh^2$ have the errors of 4.2 percent and 4.1 percent, respectively, all values being unsafe. Since in the present case only the center deflections and stresses are to be investigated and the errors are sufficiently small for engineering purposes, the case $n = 3$ is considered to be satisfactory for the final results.

The center deflections obtained by Way (reference 15), Levy (references 17 and 19), and Head and Sechler (reference 23) are plotted in figure 19 for comparison with the present results. The center membrane, bending, and total stresses are plotted in figure 20 to compare with the results by Levy (references 17 and 19). It is seen from these results that the center deflections are in good agreement with test results from the California Institute of Technology up to $\frac{pa^4}{Eh^4} = 120$. The theoretical results seem to be too low at higher pressures.

It is interesting to note that the test results are really for clamped-edge plates. The clamping effect seems to be only local, and at the center of the plate the plate behaves just as though it were simply supported; that is, the plate is free to rotate about its edges.

From the point of view of the engineer designing the plate, the total stresses at the center of the edges are still much larger in the case of clamped edges than in all the other cases; hence, a design based on those stresses would give a conservative structure. The center deflections, however, would give an idea of the magnitude of the washboarding of a boat bottom while a seaplane is taxying or landing.

CONCLUSIONS

The following conclusions may be drawn from a theoretical analysis of an initially flat, rectangular plate with large deflections under either normal pressure or combined normal pressure and side thrust:

1. The large-deflection problems of rectangular plates can be solved approximately by the present method with any boundary conditions and to any degree of accuracy required. Although it is still difficult, the present method is, nevertheless, simpler than the previously used methods for giving the same degree of accuracy.

2. For the square plate considered, case $n = 3$ gives results of good accuracy, and the results are consistent with the existing theories.

3. The clamping effect of a clamped thin plate seems to be only local. At the center, the plate behaves more like a plate with simply supported edges; that is, the thin plate is approximately free to rotate about its edges.

4. The test results show that, at $\frac{pa^4}{Eh^4} > 175$ (where $\frac{pa^4}{Eh^4}$ is nondimensional form for normal pressure), all the existing solutions of the differential equations give unsafe results for center deflection for a square plate. This conclusion perhaps suggests the range in which the differential equations may be applied.

5. The present results of the center deflections and membrane stresses give good agreement with the test results when $\frac{pa^4}{Eh^4} < 120$.

Massachusetts Institute of Technology
    Cambridge, Mass., March 4, 1946

APPENDIX A

BRIEF DESCRIPTION OF RELAXATION METHOD

The idea behind the treatment by the relaxation method is essentially just the same as that by Cross' method of moment distribution in the case of bending of continuous beams. It seems, therefore, easiest to explain the relaxation method by a comparison with the moment-distribution method, since the latter is well accepted and is familiar to most structural engineers.

The redundant beam as shown in figure 21(a) is now examined. The procedure for obtaining the redundant support moments by the moment-distribution method is well known. The first step in the moment-distribution analysis is to assume that the slope at each of the four supports is zero. By this assumption, the end moments at A, B, C, and D can be found without difficulty. The result is shown in figure 21(b). Here the boundary conditions at A and B are satisfied, and the principle of continuity is also satisfied. The condition of equilibrium, however, is not satisfied, since there are unbalanced moments at B and C. The moment-distribution method now offers a procedure to balance these unbalanced moments by a relaxation based on consistent deformations. The analysis by the relaxation method, in this case, would be essentially the same. The moments at A, B, C, and D are assumed to satisfy the boundary conditions and the condition of continuity. The unbalanced moments at B and C are then distributed by the relaxation based on consistent deformations. The difference lies in that the relaxation method offers more freedom in assuming the end moments and therefore could make the convergence of the operations more rapid. On the other hand, however, it becomes difficult to assume these values.

The method of moment distributions applies only to redundant structures, but the application of the relaxation method extends much further, and its application to the partial differential equations has brought the study of engineering sciences into a new era because the boundary conditions are now no longer difficult to be described and to be satisfied.

The procedure can be illustrated by a study of the small-deflection theory of thin plates. Letting $w = \dfrac{w'}{p}$, where w' and p are the nondimensional deflection and pressure, respectively, gives the following equilibrium equation in terms of the finite difference

$$\Delta_x^{\,4} w + 2\Delta_{xy}^{\,2} w + \Delta_y^{\,4} w = 12(1 - \mu^2)(\Delta l)^4 \tag{A1}$$

In order to solve the problem, the domain to be investigated is drawn and the net points chosen. Values of w are assumed to satisfy the boundary conditions and are then written adjacent to each point of the net. From these values of w, the residuals Q at points (m,n) are computed and recorded as follows:

$$Q_{m,n} = 20w_{m,n} - 8\left(w_{m+1,n} + w_{m-1,n} + w_{m,n+1} + w_{m,n-1}\right)$$

$$+ 2\left(w_{m+1,n+1} + w_{m+1,n-1} + w_{m-1,n+1} + w_{m-1,n-1}\right)$$

$$+ \left(w_{m+2,n} + w_{m-2,n} + w_{m,n+2} + w_{m,n-2}\right)$$

$$- 12(1 - \mu^2)(\Delta l)^4 \tag{A2}$$

The residuals Q thus computed can be thought of as an unbalanced force which must be removed from the system. Now, instead of setting up a specific iteration process, it is merely observed that if the deflection at one point (m,n) is altered, all others remaining fixed, the residuals will change according to the pattern of figure 4, the relaxation pattern. Each change of w at any point effects a redistribution of the residuals Q among the net points, and such changes of w are desired as will move all the unbalanced forces to the boundary.

For a simply supported plate, the deflection and bending moments are zero along the edges. Equation (A1) can be written as

$$\nabla^2\left(\nabla^2 w\right) = p$$

Letting $\nabla^2 w = M$ makes possible the formulation of the boundary-value problem as follows:

$$\left.\begin{array}{l} \nabla^2 M = p \\[2em] M = 0 \quad \text{along the four edges} \end{array}\right\} \tag{A3}$$

and

$$\left.\begin{array}{l} \nabla^2 w = M \\[2em] w = 0 \quad \text{along the four edges} \end{array}\right\} \tag{A4}$$

The problems can now be solved in two steps, that is, first, by use of equation (A3) and then by use of equation (A4). This transformatio: greatly reduces the labor required in applying the relaxation method because the relaxation pattern of the harmonic or Laplacian type is much simpler than that of the biharmonic type.

As an example, the boundary-value problem is solved when the plate is a square one. The process is considered with $n = 4$. From the previous results as found from the calculations with $n = 3$, the values of w at all the net points can be assumed. By equation (A4)

$$M_{m,n} = w_{m+1,n} + w_{m-1,n} + w_{m,n+1} + w_{m,n-1} - 4w_{m,n} \qquad (A5)$$

The values of $M_{m,n}$ are then recorded at the right of the corresponding net point, and the residuals

$$Q_{m,n} = M_{m+1,n} + M_{m-1,n} + M_{m,n+1} + M_{m,n-1} - 4M_{m,n} - 12(1 - \mu^2)(\Delta l)^4 \qquad (A6)$$

are computed and are recorded at the left of these net points. The results are shown in figures 22(a) and 22(b). For example,

$$M_0 = 4w_1 - 4w_0 = 4(0.0406) - 4(0.0437)$$

$$= -0.0124$$

$$M_4 = w_2 + w_3 + w_5 + w_7 - 4w_4$$

$$= 0.0377 + 0.0316 + 0.0231 + 0.0163 - 4(0.0295)$$

$$= -0.0093$$

$$Q_0 = 4M_1 - 4M_0 - 0.002637$$

$$= 4(-0.0117) - 4(-0.0124) - 0.002637$$

$$= 0.000163$$

$$Q_4 = M_2 + M_3 + M_5 + M_7 - 4M_4 - 0.002637$$

$$= -0.0106 - 0.0093 - 0.0078 - 0.0064 - 4(-0.0093) - 0.002637$$

$$= 0.001463$$

where $0.002637 = 12(1 - \mu^2)(\Delta l)^4$, since $\mu^2 = 0.1$ and $\Delta l = \frac{1}{8}$.

The largest counterbalanced M occurs in the vicinity of the greatest deviation of the assumed values from the correct solution; so changes are first made at this point. An examination of figure 22(b) shows that the greatest residual occurs at point 2. Since

$$Q_2 = 2M_1 + 2M_4 - 4M_2 - 0.002637$$

a change of M_2 would change Q_2 by an amount equal to four times $(-\Delta M_2)$. Mathematically,

$$\Delta Q_2 = -4\Delta M_2$$

where Δ denotes the amount of change. Adding -0.0004 to M_2 while assuming all the other values of M to remain unchanged gives $\Delta Q_2 = 0.0016$, and Q_2 is now equal to -0.000637. If a nomenclature similar to that in the method of moment distribution is used, this process can be called balancing the unbalanced Q. A symbol (b1) is put at the side of the value to indicate the first balancing. Now it is observed that

$$Q_1 = M_0 + 2M_2 + M_3 - 4M_1 - 0.002637$$

and

$$Q_4 = M_2 + M_3 + M_5 + M_7 - 4M_4 - 0.002637$$

A change of M_2 with all the other M's fixed would change Q_1 and Q_4 by the relations as follows:

$$\Delta Q_1 = 2\Delta M_2$$

$$\Delta Q_4 = \Delta M_2$$

Now, by relaxing the nets,

$$\Delta Q_1 = 2(-0.0004) = -0.0008$$

$$\Delta Q_4 = -0.0004$$

and

$$Q_1 = 0.001263 - 0.0008 = 0.000463$$

$$Q_4 = 0.001463 - 0.0004 = 0.001063$$

These operations may be called carrying-over and be denoted by (cl).

The whole process consists of 20 balancing and carrying-over operations by similar calculations. The detailed operations of the computations are shown in figure 22(b). After the values of M's are computed, the residuals are computed as follows:

$$Q_{m,n}{}' = W_{m+1,n} + W_{m-1,n} + W_{m,n+1} + W_{m,n-1} - 4W_{m,n} - M_{m,n}$$

The values of w may be determined by a similar series of calculations
The detailed operations and computations are shown in figure 22(a).
The whole process consists of 11 balancing and carrying-over operations
The center deflection ratio thus obtained is, for $\mu = 0.316228$,

$$w_0 = 0.043790p$$

For $\mu = 0.3$,

$$w_0 = 0.043790 \times \frac{0.91}{0.9}p$$

$$= 0.0443p$$

which checks exactly with the exact analytical solution.

For thin plates with clamped edges, the boundary conditions are

$$w = 0$$

$$\frac{\partial w}{\partial x} = 0, \quad \text{along} \quad x = \pm\frac{1}{2}$$

$$\frac{\partial w}{\partial y} = 0, \quad \text{along} \quad y = \pm\frac{1}{2}$$

The relaxation pattern of the biharmonic type must be used in this case. Although the pattern is more complicated, the process is essentially the same.

After the essential idea of the relaxation method is grasped, other problems may be solved by rather obvious steps. It may be noted that no question of convergence can occur in the general relaxation process since no specific instructions are given. If, after some steps, the residuals get worse, the intelligent computer makes changes in the opposite direction. These remarks, however, oversimplify the problem somewhat because of two facts: first, the computer may become confused as to whether the residuals are really better, and, secondly, there is always a question of whether a solution with zero residuals exists.

REFERENCES

1. Föppl, A.: Vorlesungen über technische Mechanik. Bd. 5, (Leipzig),
   1907, p. 24.

2. Von Kármán, Th.: Festigkeits probleme in Machinenbau. vol. IV of
   Encyk. der Math. Wiss, 1910, p. 349.

3. Hencky, H.: Über den Spannungoznetand in kreisrunden Platten mit
   verschwindender Biegungs-steifigkeit. Zeitschr. für Math. und Phys.,
   vol. 63, 1915, p. 311.

4. Hencky, H.: Die Berenchnung dunner rechteckiger Platten mit
   verschwindender Biegungs-steifigkeit. Z.f.a.M.M., vol. 1, 1921,
   pp. 81 and 423.

5. Kaiser, R.: Rechnerische und experimentelle Ermittlung der
   Durchbiegungen und Spannungen von quadratischen Platten bei freier
   Auflagerung an den Randern, gleichmassig verteilter Last und grossen
   Ausbiegungen. Z.f.a.M.M., Bd. 16, Heft 2, April 1936, pp. 73-98.

6. Nadai, A.: Die Elastische Platten. Julius Springer (Berlin), 1925, p. 288.

7. Timoshenko, S.: Vibration Problems in Engineering. D. Van Nostrand Co.,
   Inc., 1928, p. 317.

8. Way, S.: Bending of Circular Plates with Large Deflection. A.S.M.E.
   Trans. (A.P.M. - 56 - 12), vol. 56, no. 8. Aug. 1934, pp. 627-636.

9. Federhofer, K.: Tragfahigkeit der über die Beulgrenze belastetan
   Kreisplatte. Forschung auf dem Gebiet des Ingenieurwesens,
   vol. 11, May-June 1940, pp. 97-107.

10. Friedrichs, K. O., and Stoker, J. J.: The Non-Linear Boundary Value
    Problem of the Buckled Plate. Proc. Nat. Acad. Sci., vol. 25,
    no. 10, 1939, pp. 535-540.

11. Friedrichs, K. O., and Stoker, J. J.: The Non-Linear Boundary
    Value Problem of the Buckled Plate. Am. Jour. Math., vol. LXIII,
    no. 4, Oct. 1941.

12. Friedrichs, K. O., and Stoker, J. J.: Buckling of the Circular
    Plate beyond the Critical Thrust. Jour. Appl. Mech., March 1942,
    pp. A7-A14.

13. Boobnoff, J.: Theory of Structures of Ships. (St. Petersburg)
    vol. II, 1914.

14. Prescott, J.: Applied Elasticity. Longmans, Green, and Co.
    (London), 1924, p. 474.

15. Way, S.: Uniformly Loaded, Clamped, Rectangular Plates with Large Deflections. Proc. Fifth Int. Cong. Appl. Mech., John Wiley & Sons, Inc., 1939, pp. 123-128.

16. Green, J. R., and Southwell, R. V.: Relaxation Methods Applied to Engineering Problems, VIII A. Problems Relating to Large Transverse Displacements of Thin Elastic Plates. Phil. Trans. Roy. Soc. (London), ser. C, vol. I, no. 6, Feb. 1944, pp. 137-176.

17. Levy, Samuel: Square Plate with Clamped Edges under Normal Pressure Producing Large Deflections. NACA TN No. 847, 1942.

18. Levy, Samuel, and Greenman, Samuel: Bending with Large Deflection of a Clamped Rectangular Plate with Length-Width Ratio of 1.5 under Normal Pressure. NACA TN No. 853, 1942.

19. Levy, Samuel: Bending of Rectangular Plates with Large Deflections. NACA TN No. 846, 1942.

20. Niles, Alfred S., and Newell, Joseph S.: Airplane Structures. Vol. I Third Ed., John Wiley & Sons, Inc., 1943, p. 329.

21. McPherson, Albert E., Ramberg, Walter, and Levy, Samuel: Normal-Pressure Tests of Circular Plates with Clamped Edges. NACA TN No. 848, 1942.

22. Ramberg, Walter, McPherson, Albert E., and Levy, Samuel: Normal-Pressure Tests of Rectangular Plates. NACA TN No. 849, 1942.

23. Head, Richard M., and Sechler, Ernest E.: Normal-Pressure Tests on Unstiffened Flat Plates. NACA TN No. 943, 1944.

24. Crout, P. D.: A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients. Trans. A.I.E.E., vol. 60, 1941.

25. Mesnager, A.: Calcul elementaire rigoureux des plaques rectangulaires. Inst. Assoc. for Bridge and Structural Eng., 1932, pp. 329-336.

26. Gall, H. W.: Compressive Strength of Stiffened Sheets of Aluminum Alloy: Thesis, M.I.T., 1930.

27. Timoshenko, S.: Theory of Plates and Shells. McGraw-Hill Book Co., Inc., 1940, p. 133.

TABLE 1.- SOLUTIONS OF EQUATION (71); GIVEN MATRIX

F_0	F_1	F_2	F_3	F_4	K
20.00000000	-32.00000000	8.00000000	4.00000000	0,	K_0
-8.00000000	24.00000000	-16.00000000	-6.63245600	6.632456000	K_1
2.00000000	-16.00000000	20.00000000	4.632456000	-13.26491200	K_2
-2.00000000	-4.00000000	4.00000000	-.632456000	2.63245600	S_1
1.00000000	0	-6.00000000	2.31622800	1.36754400	S_2

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 2.- SOLUTIONS OF EQUATION (71); AUXILIARY MATRIX

20.00000000	-1.600000000	0.400000000	0.200000000	0
-8.00000000	11.20000000	-1.14285714	-.44932643	.59218357
2.00000000	-12.80000000	4.57142861	-.33226425	-1.24358549
-2.00000000	-7.200000000	-3.42857141	-4.60679800	-.57142858
1.00000000	1.600000000	-4.57142858	1.31622800	-4.51278167

Last column

K_0	K_1	K_2	S_1	S_2
0.05000000				
.03571429	0.08928571			
.07812501	.24999999	0.21875000		
-.13566908	-.32560575	-.16280288	-0.21707051	
-.09496836	-.31656115	-.26907699	-.06331223	-0.22159282

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 3.- SOLUTIONS OF EQUATION (71); FINAL MATRIX

K_0	K_1	K_2	S_1	S_2
-0.04870254	-0.26569616	-0.22511074	-0.30411392	-0.30952531
-.11120254	-.30736283	-.23552741	-.26244725	-.28869198
-.10308545	-.31196199	-.22105220	-.16287974	-.31764240
-.18993672	-.50649784	-.31656116	-.25324893	-.12662447
-.09496836	-.31656115	-.26907699	-.06331223	-.22159282

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 4.- SOLUTIONS OF EQUATION (61)

	K_0	K_1	K_2	K_3	K_4	K_5
$F_0 =$	-0.118035	-0.622023	-0.632001	-0.496545	-0.904377	-0.288561
$F_1 =$	-.199285	-.714880	-.673668	-.514402	-.923424	-.290942
$F_2 =$	-.224028	-.787231	-.673033	-.525863	-.916444	-.288867
$F_3 =$	-.306047	-1.034464	-.883271	-.580764	-1.032622	-.306995
$F_4 =$	-.261272	-.927438	-.818231	-.555182	-.961845	-.293340
$F_5 =$	-.186116	-.680999	-.620292	-.462889	-.830524	-.254089
$F_6 =$	-.427358	-1.410958	-1.139620	-.841148	-1.302423	-.352740
$F_7 =$	-.303899	-1.107060	-1.012996	-.651212	-1.201124	-.340077
$F_8 =$	-.123459	-.466702	-.443186	-.352740	-.680154	-.274956

S_1	S_2	S_3
-0.292423	-0.439791	-0.182625
-.274566	-.420743	-.177863
-.233883	-.413113	-.182012
-.266646	-.340766	-.145756
-.175344	-.353101	-.173066
-.092310	-.206823	-.251569
-.298472	-.217071	-.054268
-.108535	-.318370	-.079593
-.027134	-.079593	-.209835

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 5.- SOLUTIONS OF EQUATION (62); GIVEN MATRIX

w_0	w_1	w_2	w_3	w_4	w_5
25.995771	-38.246029	8.250258	4.000000	0	0
-9.686636	31.006139	-18.632867	-10.281802	6.595166	0
2.000000	-18.382610	28.241968	4.000000	-21.336107	3.476749
1.000000	-10.281802	4.000000	24.752966	-17.189361	2.000000
0	3.000000	-9.632096	-8.297097	26.941238	-9.379948
0	0	2.000000	2.000000	-16.200297	17.832449

p'	Check column
0.833333	0.833333
.833333	-.166667
.833333	-1.166667
.833333	5.115135
.833333	3.465429
.833333	6.465485

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 6.- SOLUTIONS OF EQUATION (62); AUXILIARY MATRIX

25.995771	-1.4712404	0.3173692	0.1538712	0	0
-9.686636	16.7547688	-.9286089	-.5247048	.3936292	0
2.000000	-15.4401292	13.2693882	-.3322875	-1.1498964	.2620128
1.000000	-8.8105616	-4.4989351	18.4812109	-1.0223668	.1720006
0	3.000000	-6.8462693	-8.9979123	8.6886832	-.6949835
0	0	2.000000	2.6645750	-11.1763312	9.0827491

0.0320565	0.0320565
.0682703	.0085858
.1374082	-.0827630
.1093525	.2589862
.2938535	.5988697
.3909987	1.3909983

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 7.- CENTER DEFLECTIONS

$\dfrac{pa^4}{Eh^4}$	w_0/h			
	$n = 1$	$n = 2$	$n = 3$	$n = 4$
0	0	0	0	------
12.5	.3888	.4062	.4055	------
25	.5844	.6092	.6083	------
50	.8184	.8474	.8460	------
75	.9757	1.0052	1.0031	------
100	1.0980	1.1269	1.1240	1.1250
150	1.2888	1.3145	1.3104	------
200	1.4376	1.4616	1.4557	------
250	1.5623	1.5844	1.5770	------

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 8.- DEFLECTIONS AT VARIOUS POINTS

$$\left[n = 2 \right]$$

pa^4/Eh^4	w_0/h	w_1/h	w_2/h
0	0	0	0
12.5	.4062	.2980	.2198
25	.6092	.4508	.3363
50	.8474	.6332	.4791
75	1.0052	.7555	.5766
100	1.1269	.8502	.6528
150	1.3145	.9966	.7713
200	1.4616	1.1116	.8648
250	1.5844	1.2076 .	.9431

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 9.- DEFLECTIONS AT VARIOUS POINTS

$$\left[n = 3 \right]$$

pa^4/Eh^4	w_0/h	w_1/h	w_2/h	w_3/h	w_4/h	w_5/h
0	0	0	0	0	0	0
12.5	.4055	.3564	.3136	.2139	.1890	.1159
25	.6083	.5365	.4738	.3249	.2892	.1822
50	.8460	.7494	.6650	.4592	.4131	.2711
75	1.0031	.8905	.7930	.5500	.4986	.3370
100	1.1240	.9995	.8920	.6207	.5660	.3915
150	1.3104	1.1677	1.0450	.7305	.6717	.4804
200	1.4557	1.2988	1.1641	.8164	.7551	.5531
250	1.5770	1.4081	1.2634	.8880	.8249	.6149

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 10.- MEMBRANE STRESSES

$$\left[\begin{array}{l}\text{Subscript } 0 \text{ denotes center of plate;}\\ \text{subscript } 1 \text{ denotes center of sides}\\ \left(x = \pm\frac{a}{2}\right)\end{array}\right]$$

$\dfrac{pa^4}{Eh^4}$	$\dfrac{\sigma_{x_0}{}'a^2}{Eh^2} = \dfrac{\sigma_{y_0}{}'a^2}{Eh^2}$		$\dfrac{\sigma_{y_1}{}'a^2}{Eh^2}$		$\dfrac{\sigma_{x_1}{}'a^2}{Eh^2}$	
	$n = 2$	$n = 3$	$n = 2$	$n = 3$	$n = 2$	$n = 3$
0	0	0	0	0	0	0
12.5	.6103	.6089	.3338	.3795	1.055	1.200
25	1.384	1.377	.7612	.8574	2.407	2.711
50	2.695	2.683	1.484	1.661	4.693	5.254
75	3.806	3.792	2.096	2.341	6.628	7.401
100	4.802	4.785	2.643	2.943	8.357	9.305
150	6.566	6.542	3.613	4.001	11.43	12.65
200	8.136	8.103	4.473	4.929	14.15	15.59
250	9.575	9.533	5.264	5.778	16.64	18.27

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

TABLE 11.- EXTREME-FIBER BENDING AND

TOTAL STRESSES AT CENTER OF PLATE

$\dfrac{pa^4}{Eh^4}$	Bending stresses, $\dfrac{\sigma''a^2}{Eh^2}$		Total stresses, $\dfrac{\sigma'a^2}{Eh^2} + \dfrac{\sigma''a^2}{Eh^2}$	
	$n = 2$	$n = 3$	$n = 2$	$n = 3$
0	0	0	0	0
12.5	2.530	2.582	3.140	3.191
25	3.708	3.781	5.092	5.158
50	5.010	5.087	7.705	7.770
75	5.845	5.928	9.651	9.720
100	6.475	6.554	11.277	11.339
150	7.439	7.513	14.005	14.055
200	8.191	8.261	16.327	16.364
250	8.817	8.891	18.392	18.424
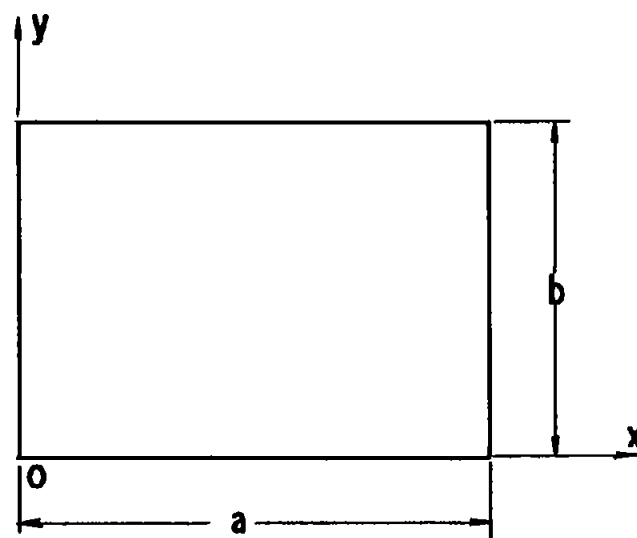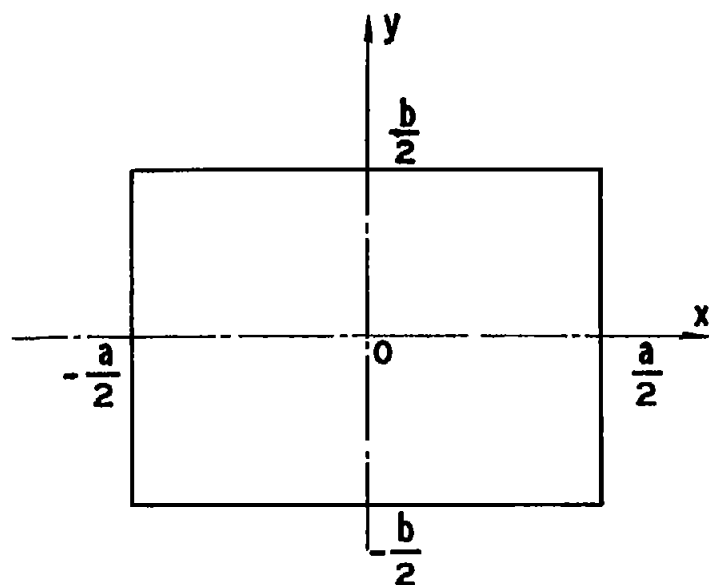
NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS
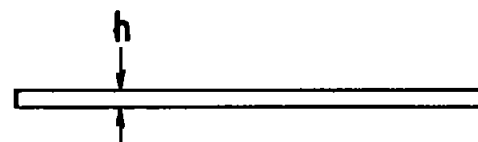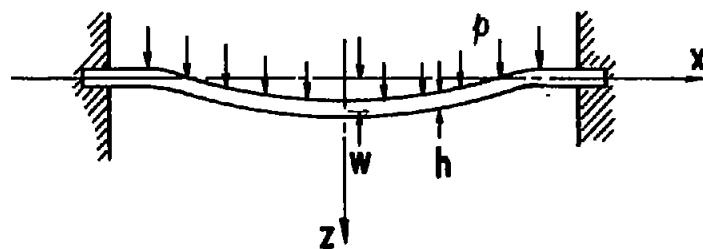
Figure 1.- Coordinate system and typical loading.

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Figure 2.- Coordinate system used by Levy
and Greenman.
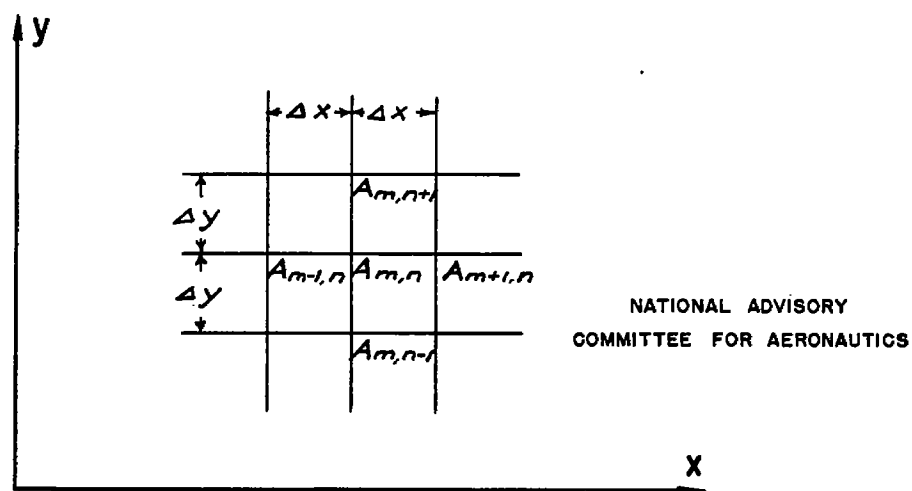
Figure 3.-   Finite-difference notation.
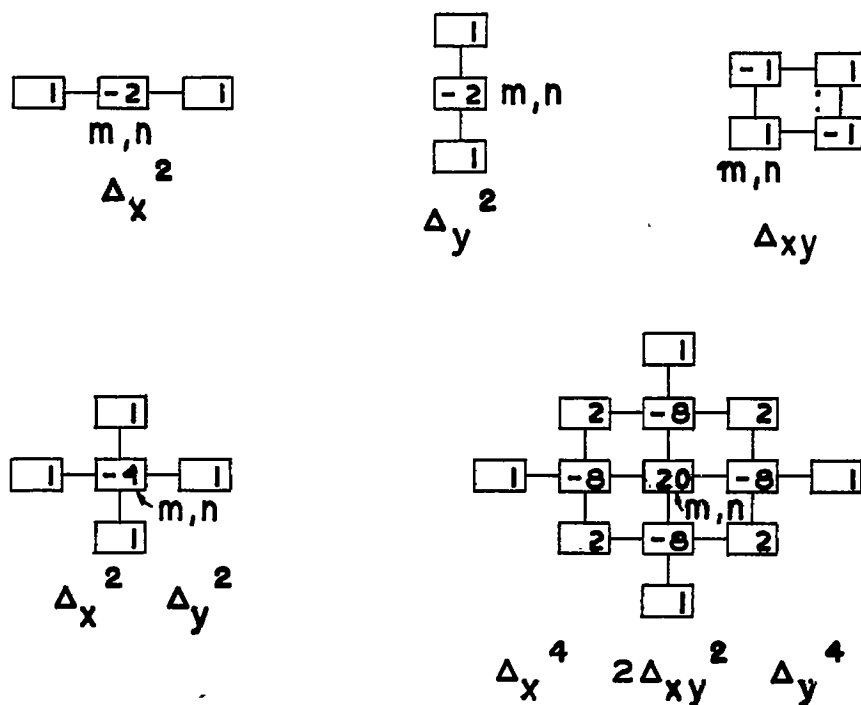
Figure 4.-   Relaxation pattern.
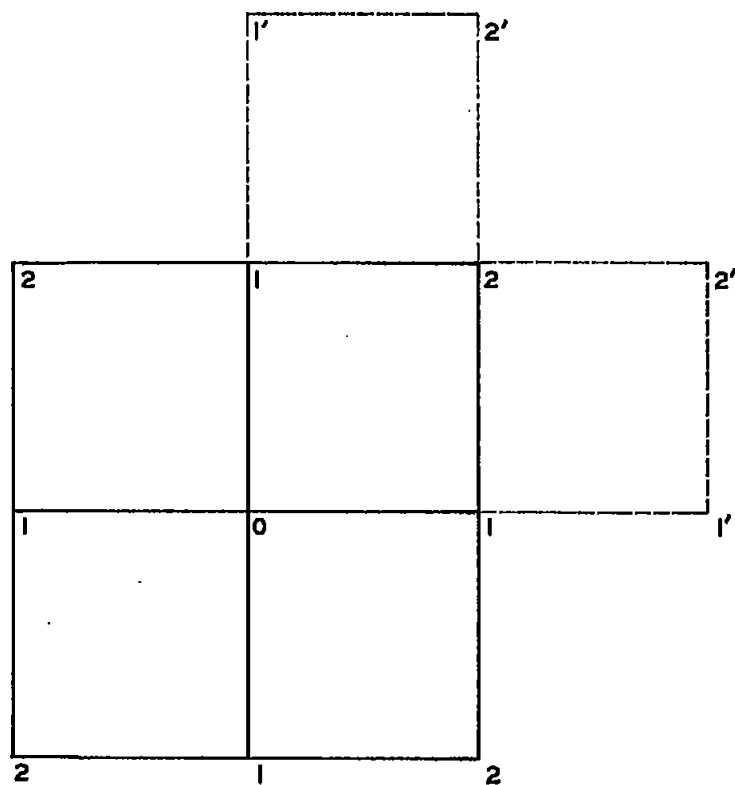
Figure 5.- Domain of problem for n = 1.
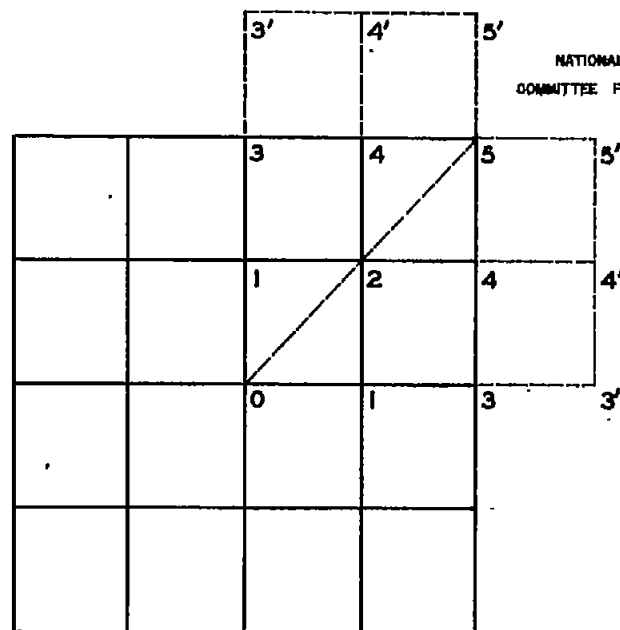
Figure 6.- Domain of problem for n = 2.
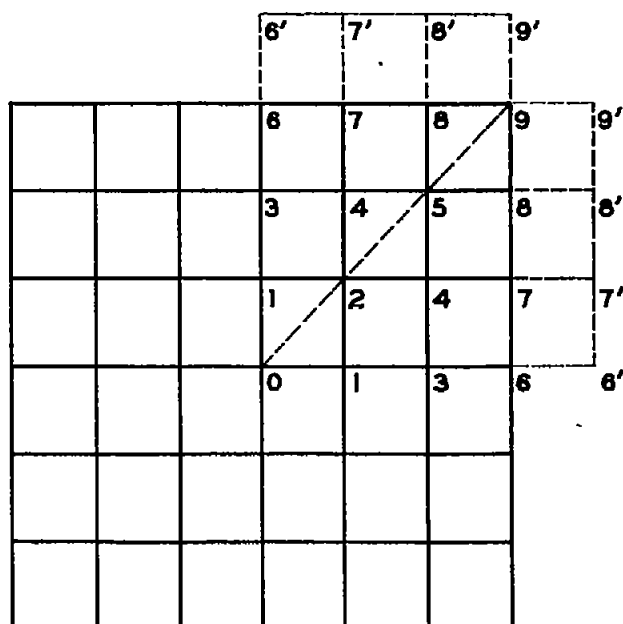
Figure 7.- Domain of problem for n = 3.
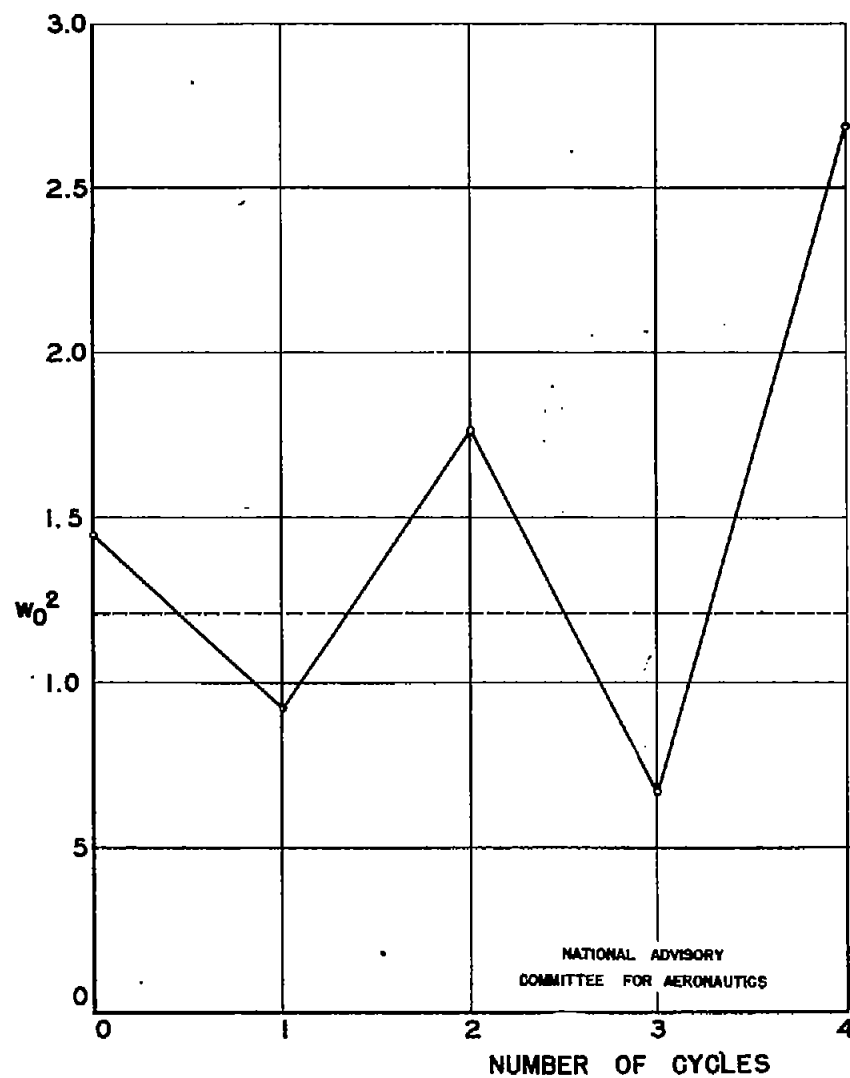
Figure 8.- Divergent values of $w_0{}^2$.
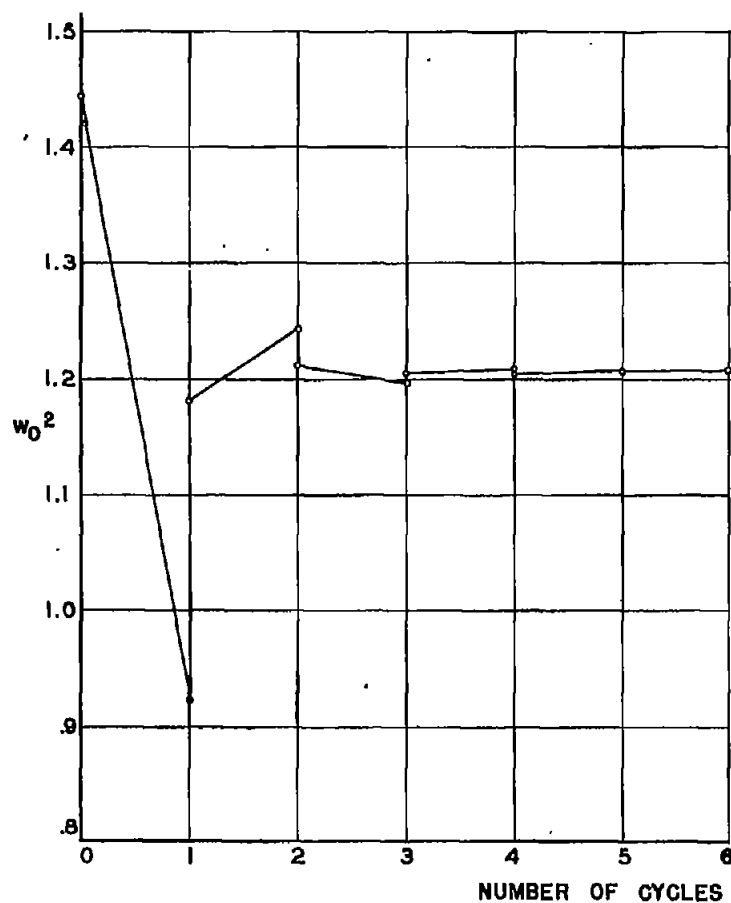
Figure 9.- Convergent values of w_0^2.
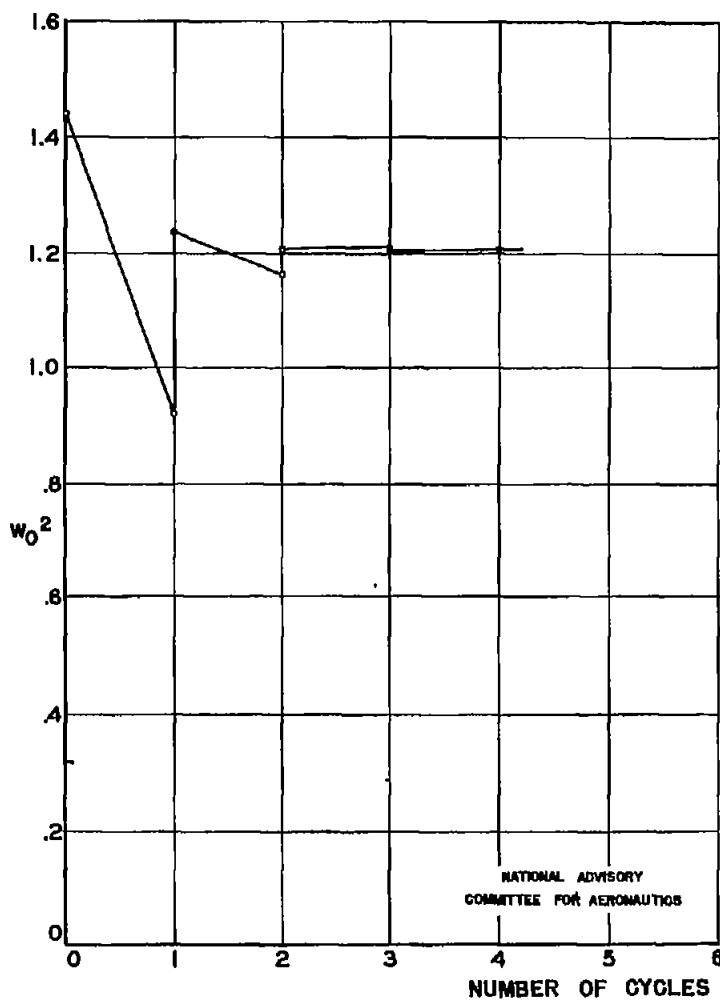
Figure 10.- Convergent values of w_0^2. Value of w_0^2 for the second cycle assumed equal to the sum of 0.6 times the assumed value for the first cycle and 0.4 times the value found from the first cycle, and so on.
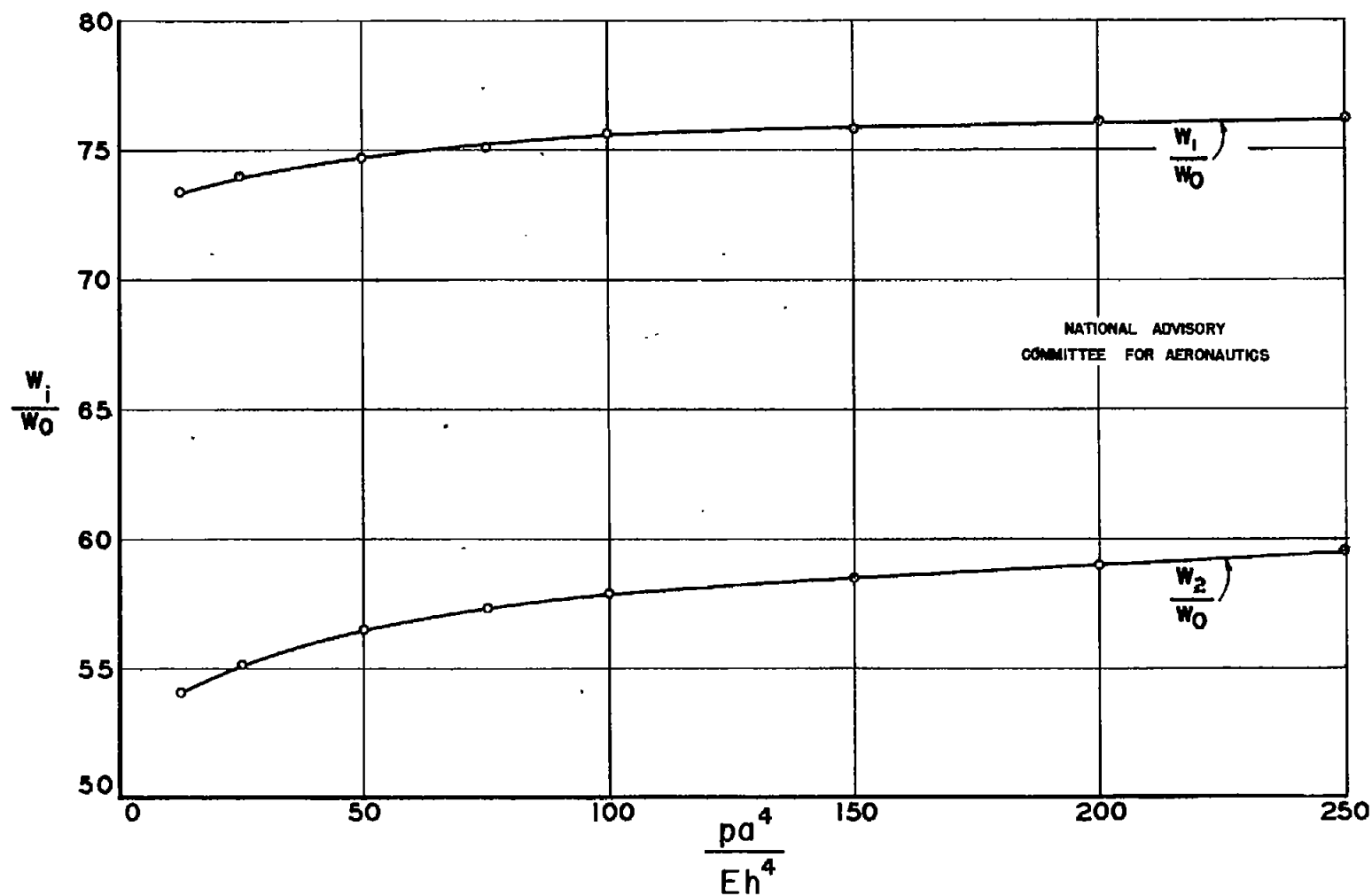
Figure 11.- Curves for $w_0 \sim p$, $w_1/w_0 \sim p$, and $w_2/w_0 \sim p$. $n = 2$.
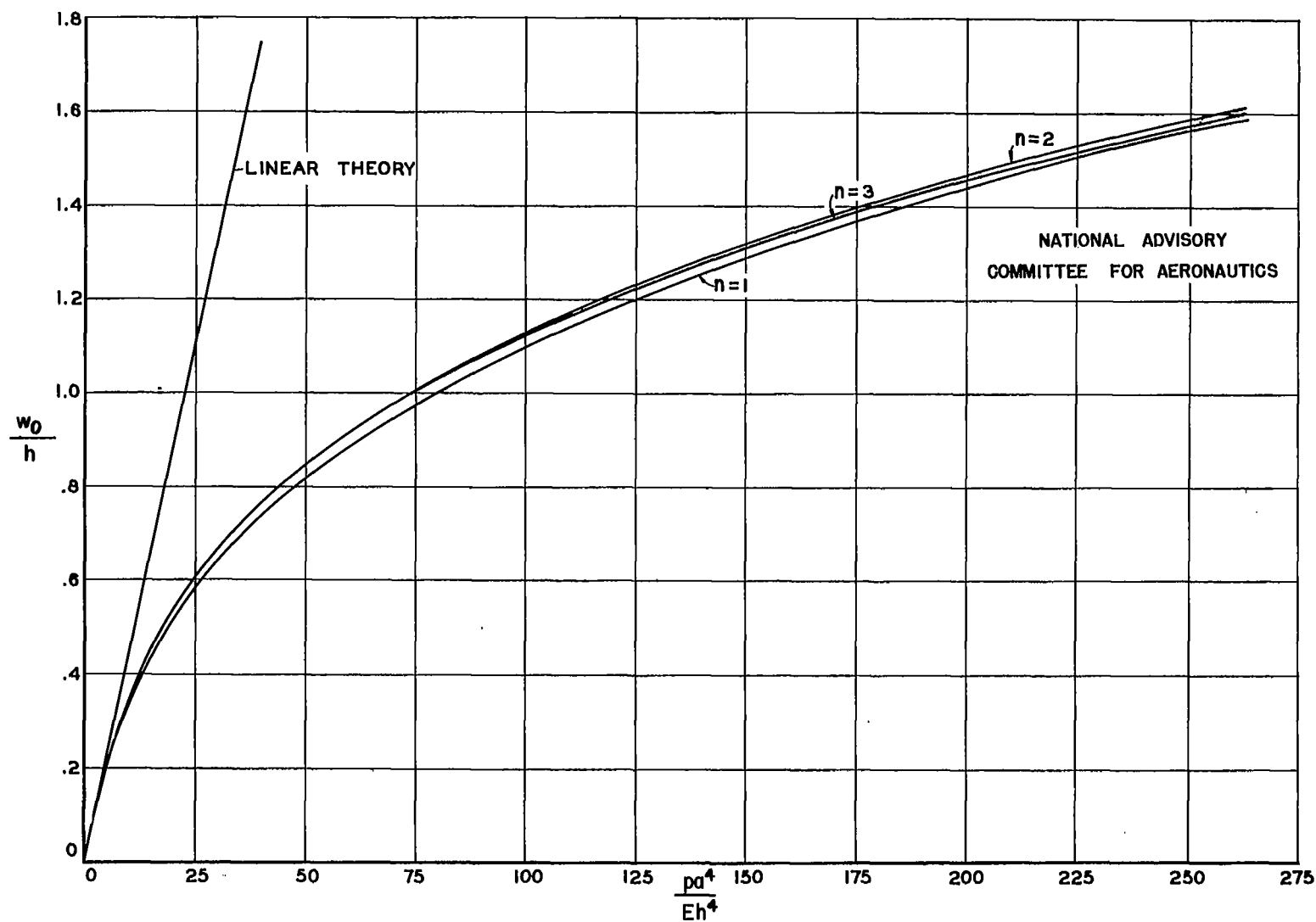
Figure 12.- Center deflections for a square plate under normal pressure p.
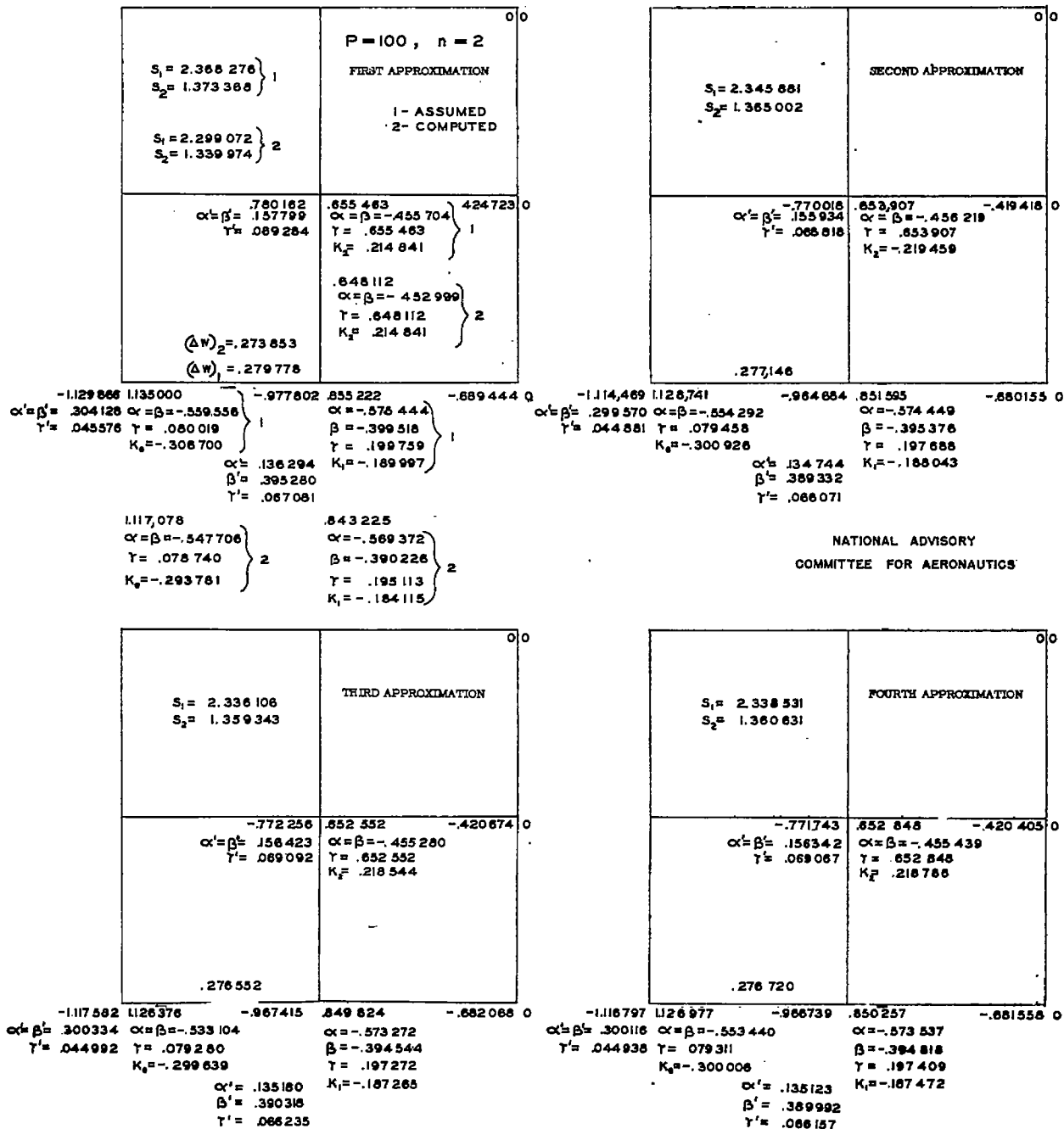
Figure 13.- Method of tabulation of α , β , γ , $(w_{n+1} - w_n)$, K, S, and F. n = 2.
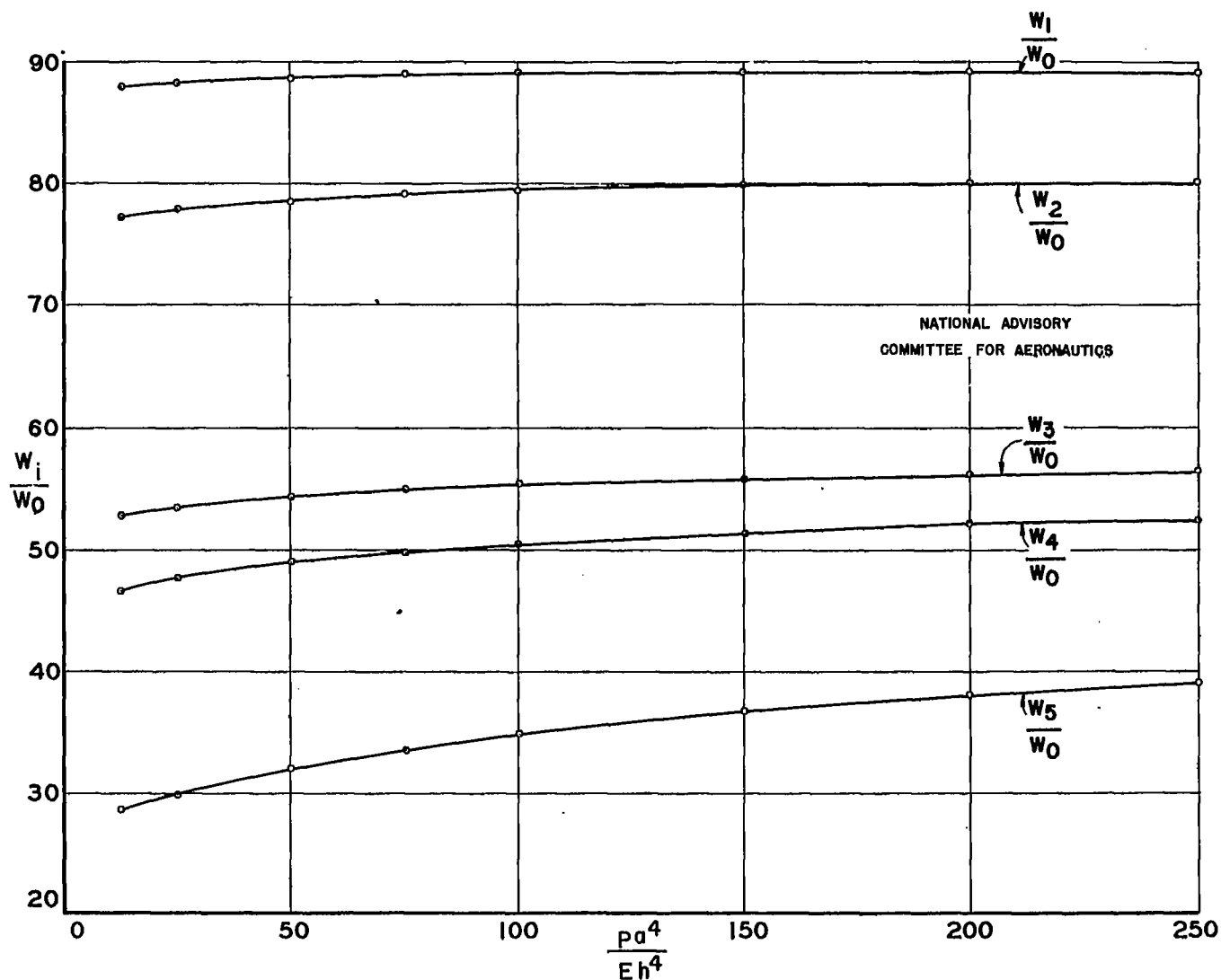
Figure 14.- Curves of $w_0 \sim p$, $w_1/w_0 \sim p$, $w_2/w_0 \sim p$, $w_3/w_0 \sim p$, $w_4/w_0 \sim p$, and $w_5/w_0 \sim p$. $n = 3$.

			0	0
$S_1 = 1.592\,696$ $S_2 = 1.282\,838$ $\Big\}$ 1 $S_3 = .548\,700$ $S_1 = 1.590\,165$ $S_2 = 1.280\,080$ $\Big\}$ 2 $S_3 = .545\,095$			1 ASSUMED 2 COMPUTED $(\Delta w)_1 = .393\,308$	

$-.50576$ $.393\,308$ $-.239\,729$ 0

$\alpha' = \beta' = .009\,273$ $\alpha = \beta = -.220\,105$
$\gamma' = -.026\,303$ $\gamma = .393\,308$
$K_5 = .106\,245$

$.390\,999$
$\alpha = \beta = -.216\,407$
$\gamma = .390\,999$
$K_5 = .106\,048$

$(\Delta w)_1 = .107\,859$ $(\Delta w)_1 = .325\,601$ $(\Delta w)_1 = .566\,511$

$-.950\,911$ $.892\,112$ $-.762\,520$ $.566\,511$ $-.546\,620$ 0

$\alpha' = \beta' = .110\,306$ $\alpha = \beta = -.217\,742$ $\alpha' = .027\,503$ $\alpha = -.240\,910$
$\gamma' = .068\,368$ $\gamma = .152\,398$ $\beta' = .151\,120$ $\beta = -.119\,329$
$K_2 = -.024\,186$ $\gamma' = .050\,132$ $\gamma = .173\,203$
$K_4 = .001\,252$

$(\Delta w)_2 = .107\,491$ $(\Delta w)_2 = .325\,874$ $(\Delta w)_2 = .174\,592$
$.891\,465$ $.565\,591$
$\alpha = \beta = -.218\,383$ $\alpha = -.239\,717$
$\gamma = .151\,282$ $\beta = -.119\,841$
$K_2 = .024\,805$ $\gamma = .174\,592$
$K_4 = .001\,754$

$(\Delta w)_1 = .124\,729$ $(\Delta w)_1 = .379\,586$ $(\Delta w)_1 = .620\,385$

$-1.095\,495$ $1.124\,700$ $-1.028\,996$ $.999\,971$ $-.868\,159$ $.620\,385$ $-.675\,850$ 0

$\alpha' = \beta' = .132\,938$ $\alpha = \beta = -.249\,458$ $\alpha' = .099\,338$ $\alpha = -.259\,857$ $\alpha' = .031\,472$ $\alpha = -.240\,799$
$\gamma' = .011\,586$ $\gamma = .016\,870$ $\beta' = .156\,170$ $\beta = -.215\,718$ $\beta' = .211\,278$ $\beta = -.107\,748$
$K_0 = -.061\,642$ $\gamma' = .027\,554$ $\gamma = .053\,985$ $\gamma' = .023\,591$ $\gamma = .053\,874$
$K_1 = -.052\,063$ $K_3 = -.023\,093$

$(\Delta w)_2 = .124\,428$ $(\Delta w)_2 = .378\,614$ $(\Delta w)_2 = .620\,342$
$1.123\,384$ $.998\,956$ $.620\,342$
$\alpha = \beta = -.248\,856$ $\alpha = -.259\,186$ $\alpha = -.241\,728$
$\gamma = .016\,937$ $\beta = -.214\,982$ $\beta = -.109\,502$
$K_0 = -.061\,642$ $\gamma = .052\,740$ $\gamma = .054\,751$
$K_1 = -.051\,864$ $K_3 = -.023\,472$

(a) First approximation.

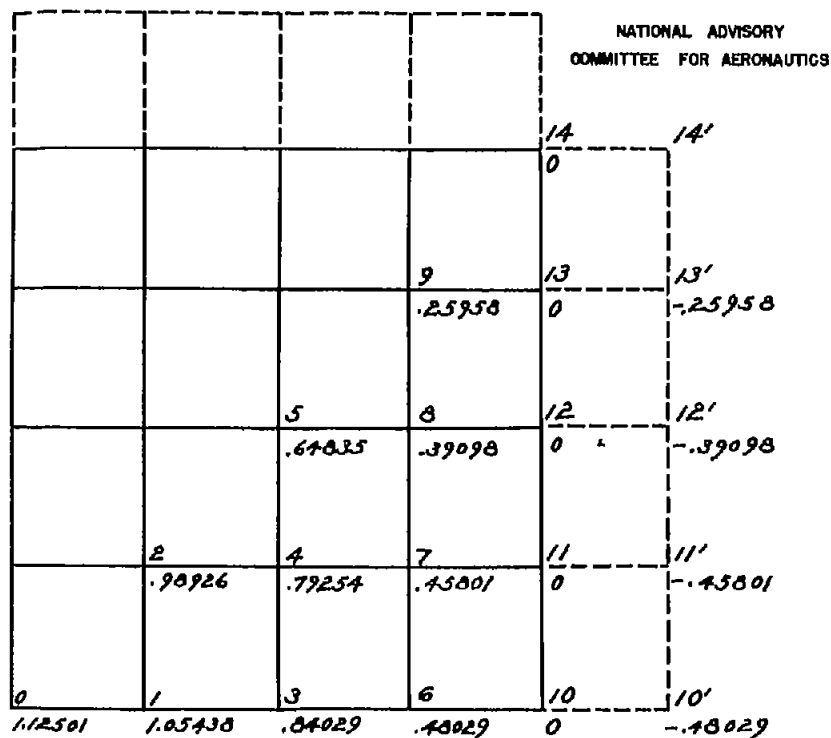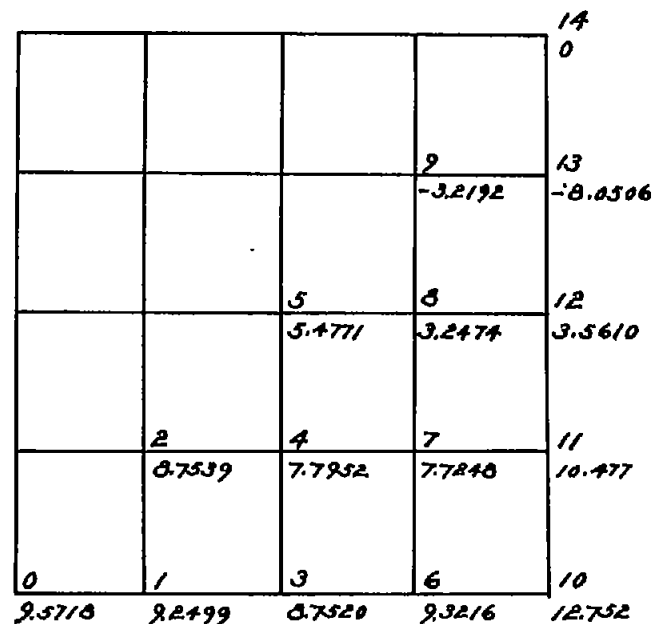Figure 15.- Results of various approximations. p = 100; n = 3.

O | O

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

$S_1 = 1.592\ 781$

$S_2 = 1.282\ 296$

$S_3 = .596\ 229$

-.504 899 | .391 465   -.239 205 | O

$\alpha'=\beta'= .009\ 113$ $\alpha=\beta=-.216\ 810$

$Y'=-.026\ 988$ $Y= .391\ 465$

$K= .106\ 208$

.107 554   .326 097

-.949 710 | .892 217   -.761 472 | .566 120   -.595 872 | O

$\alpha'=\beta'= .110\ 215$ $\alpha=\beta=-.218\ 593$ $\alpha'= .027\ 362$ $\alpha=-.240\ 023$

$Y= .068\ 340$ $Y= .151\ 942$ $\beta'= .130\ 981$ $\beta=-.119\ 908$

$K=-.024\ 826$ $Y'= .050\ 091$ $Y= .174\ 655$

$K= .001\ 724$

.124 521   .378 909

-1.094 165 | 1.124 292   -1.027 733 | .999 771   -.867 069 | .620 867   -.675 090 | O

$\alpha'=\beta'= .132\ 864$ $\alpha=\beta=-.249\ 042$ $\alpha'= .094\ 232$ $\alpha=-.254\ 383$ $\alpha'= .031\ 365$ $\alpha=-.241\ 963$

$Y'= .011\ 591$ $Y= .016\ 967$ $\beta'= .156\ 016$ $\beta=-.215\ 108$ $\beta'= .211\ 194$ $\beta=-.109\ 494$

$K=-.061\ 734$ $Y'= .027\ 574$ $Y= .052\ 807$ $Y'= .023\ 571$ $Y= .054\ 747$

$K=-.051\ 931$ $K=-.023\ 496$

(b) Second approximation.

Figure 15.- Continued.

$S_1 = 1.592\ 078$

$S_2 = 1.281\ 814$

$S_3 = .596\ 173$

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

$0 \mid 0$

$-.504\ 917 \mid .391\ 496 \qquad -.239\ 164 \mid 0$

$\alpha = \beta' = .009\ 053 \mid \alpha = \beta = -.216\ 946$

$\gamma' = -.026\ 589 \mid \gamma = .391\ 496$

$K = .106\ 204$

$.107,511 \qquad .325\ 987$

$-.919\ 937 \mid .892\ 033 \qquad -.761,617 \mid .566\ 046 \qquad -.545\ 976 \mid 0$

$\alpha' = \beta' = .110\ 265 \mid \alpha = \beta = -.218\ 976 \qquad \alpha' = .027\ 319 \mid \alpha = -.290\ 059$

$\gamma' = .068\ 380 \mid \gamma = .151\ 937 \qquad \beta' = .151\ 039 \mid \beta = -.119\ 850$

$K = -.024\ 799 \mid \gamma' = .050\ 114 \mid \gamma = .174\ 550$

$K = .001\ 697$

$.124\ 482 \qquad .378\ 798$

$-1.094\ 445 \quad 1.124\ 026 \qquad -1.027\ 392 \mid .939\ 544 \qquad -.267\ 218 \mid .620\ 746 \qquad -.675\ 217\ 0$

$\alpha' = \beta' = .132\ 906 \mid \alpha = \beta = -.848\ 964 \mid \alpha' = .039\ 260 \mid \alpha = -.259\ 316 \mid \alpha' = .031\ 347 \mid \alpha = -.291\ 978$

$\gamma' = .011\ 602 \mid \gamma = .016\ 971 \mid \beta' = .196\ 110 \mid \beta = -.215\ 022 \mid \beta' = .211\ 322 \mid \beta = -.109\ 400$

$K = -.061\ 695 \mid \gamma' = .027\ 606 \mid \gamma = .052\ 811 \mid \gamma' = .023\ 578 \mid \gamma = .054\ 700$

$K = -.051\ 894 \mid K = -.023\ 477$

(c) Third approximation.

Figure 15.- Concluded.

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

(a) Domain of problem, w-plane.

Figure 16.- Relaxation method. p = 100; n = 4.

(b) Domain of problem, T-plane.

Figure 16.- Continued.
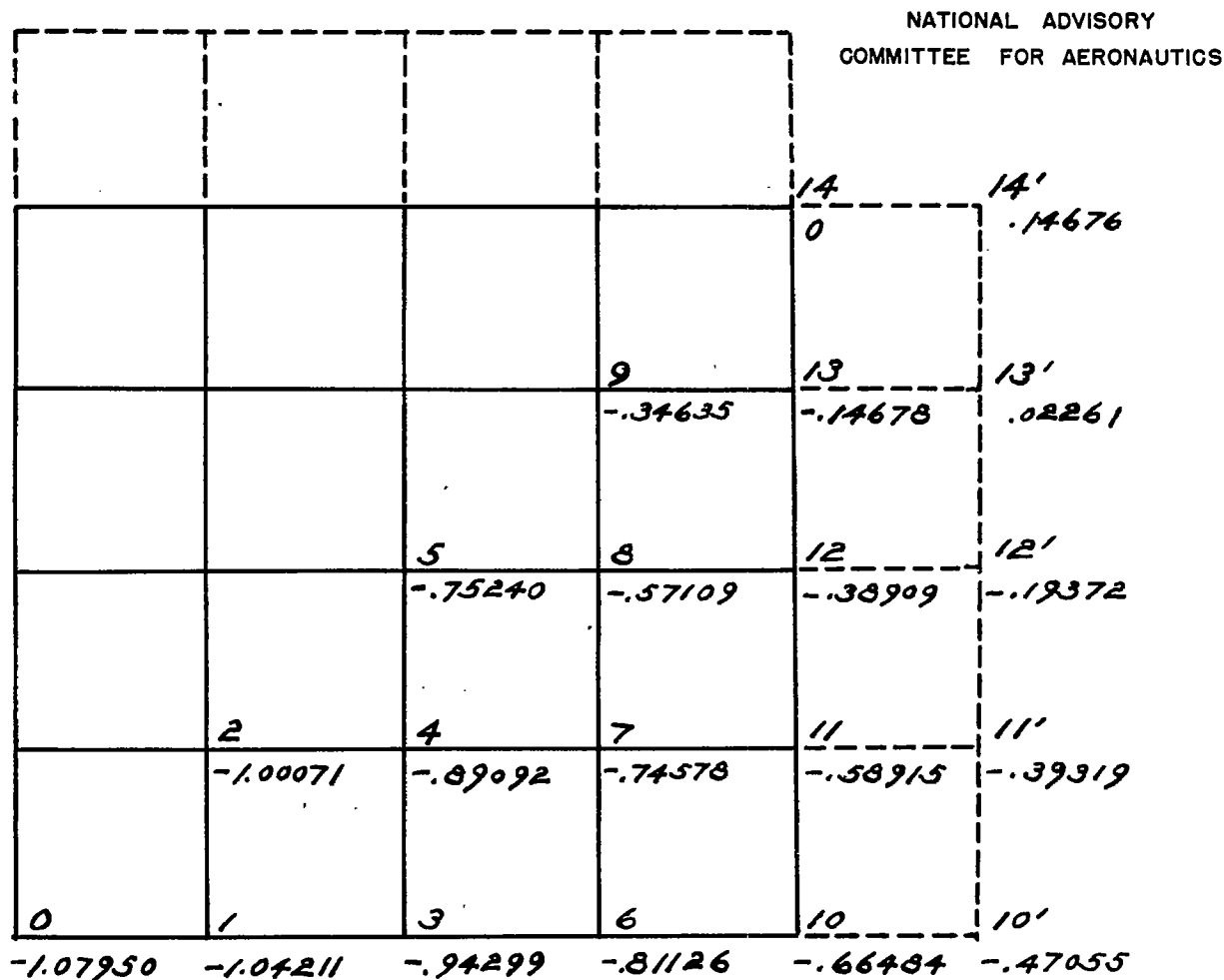
(c) Domain of problem, F-plane.

Figure 16.- Concluded.
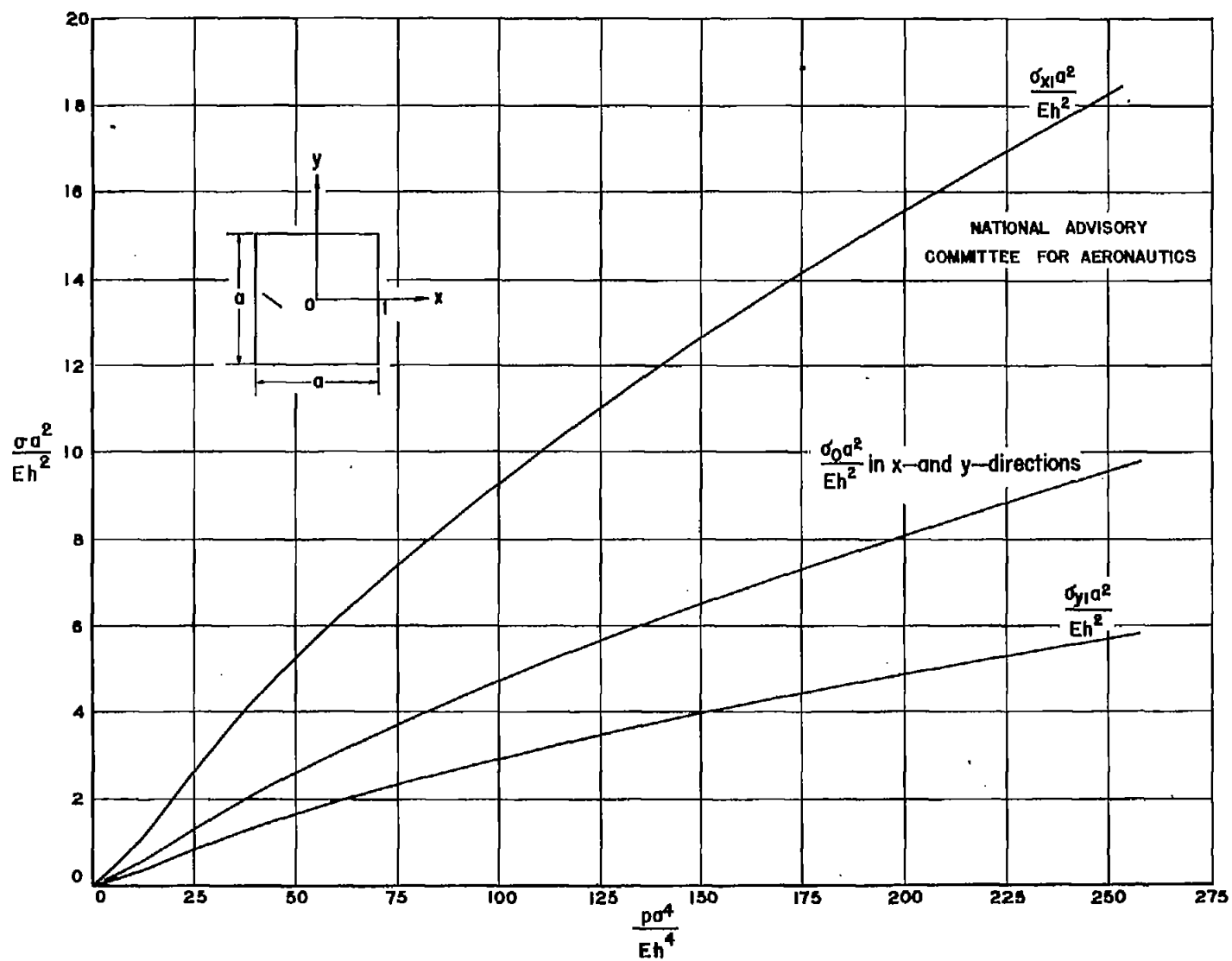
Figure 17.- Membrane stresses. n = 3.
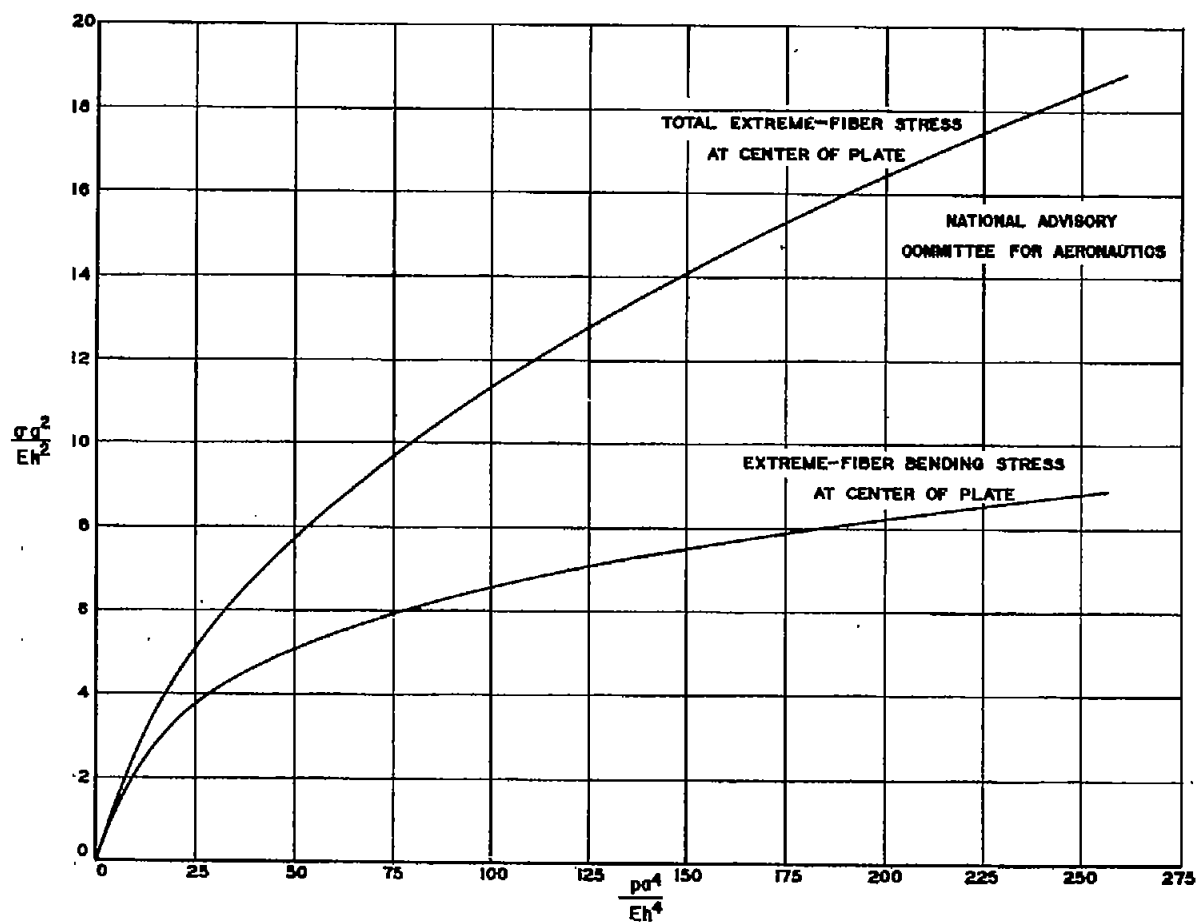
Figure 18.- Extreme-fiber total and bending stresses at center of plate.   n = 3.
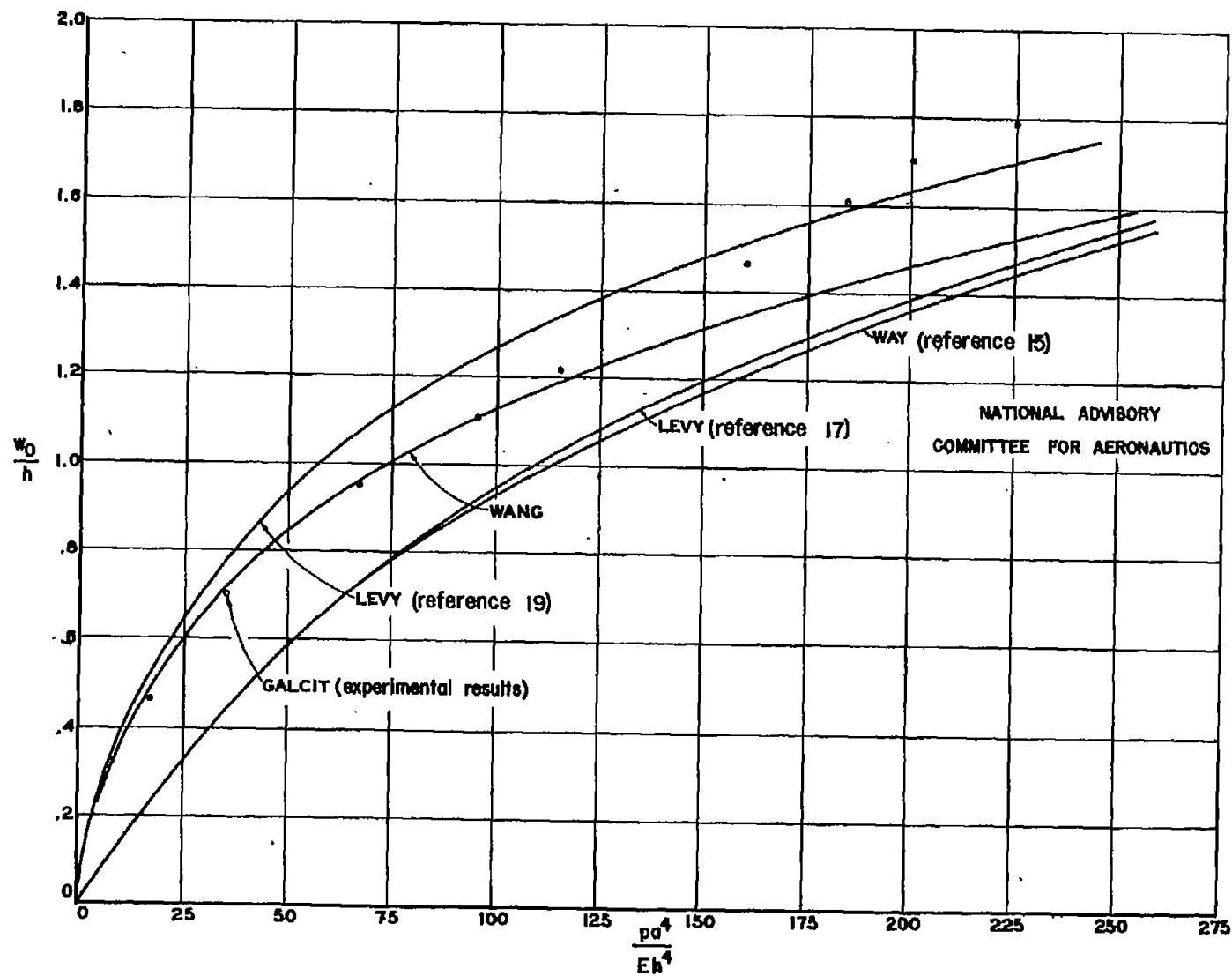
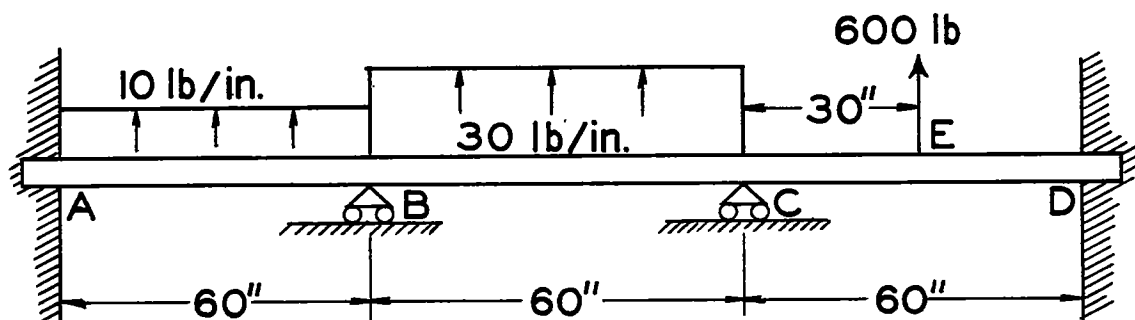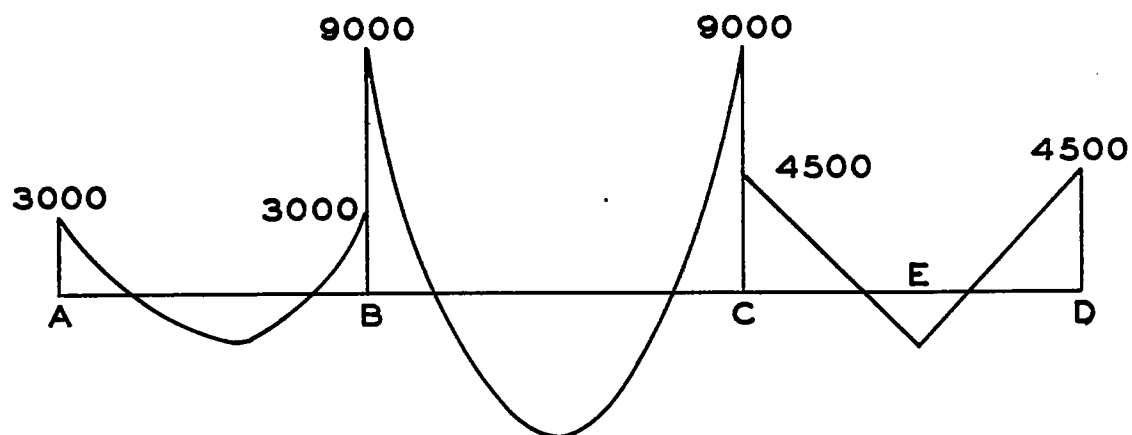Figure 19.- Center deflections as obtained by experimental and various theories.
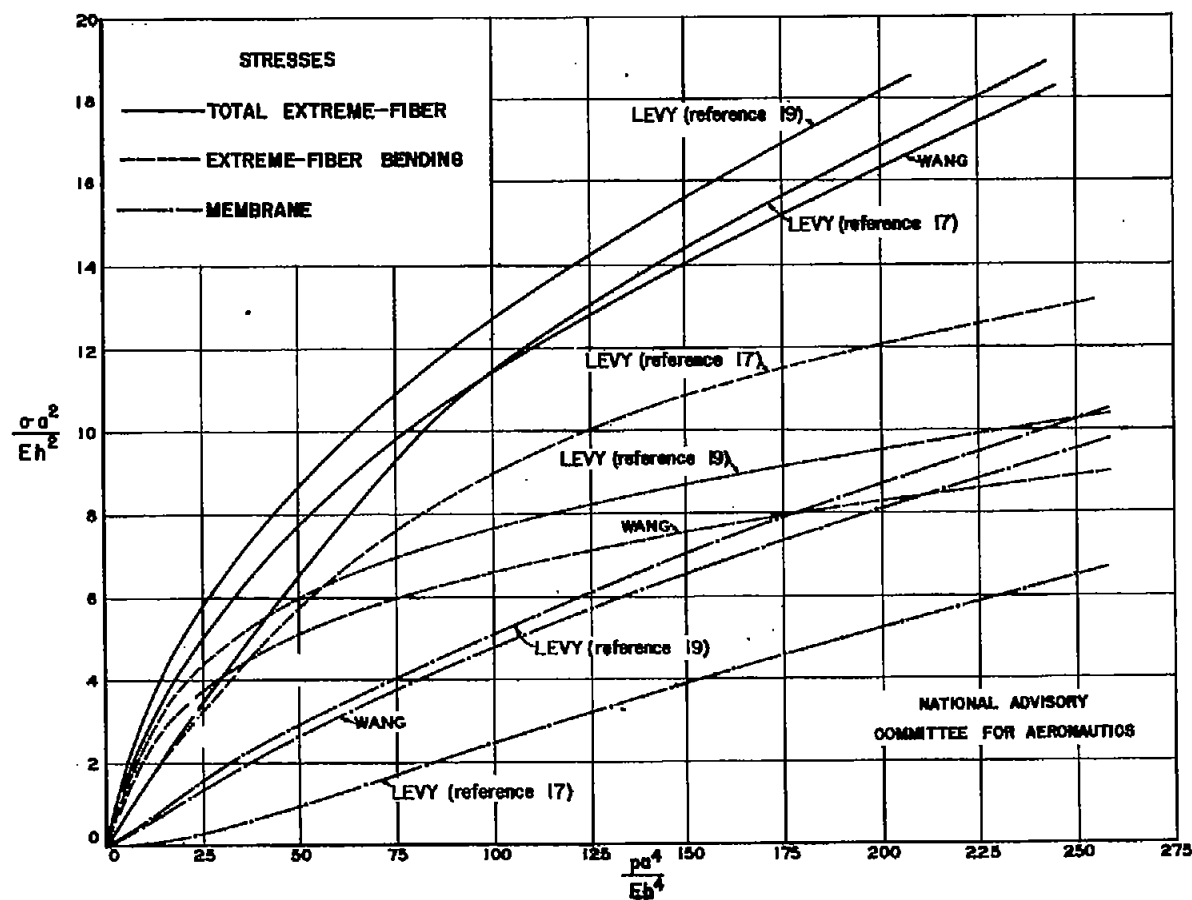
Figure 20.- Center stresses as computed from various theories.

(a)   Redundant beam.

NATIONAL   ADVISORY
COMMITTEE   FOR   AERONAUTICS

(b)   End moments for zero slope at each support.

Figure 21.-   Moment-distribution method.

(a) $\nabla^2 w = M$; $Q' = \nabla^2 w - M$.

Figure 22.-   Solution of small-deflection theory by relaxation method.

NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

(b) $\nabla^2 M = 12(1 - \mu^2)(\Delta z)^4 = 0.002637$; $Q = \nabla^2 M - 0.002637$.

Figure 22.- Concluded.